On the representation theory of $G \sim S_n$

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Abstract

In the Vershik-Okounkov approach to the complex irreducible representations of S_n and $G \sim S_n$ we parametrize the irreducible representations and their bases by spectral objects rather than combinatorial objects and then, at the end, give a bijection between the spectral and combinatorial objects. The fundamental ideas are similar in both cases but there are additional technicalities involved in the $G \sim S_n$ case. This was carried out by Pushkarev.

The present work gives a fully detailed exposition of Pushkarev's theory. For the most part we follow the original but our definition of a Gelfand-Tsetlin subspace, based on a multiplicity free chain of subgroups, is slightly different and leads to a more natural development of the theory. We also work out in detail an example, the generalized Johnson scheme, from this viewpoint.

1 Introduction

Let G be a finite group. The symmetric group S_n acts on $G^n = G \times \cdots \times G$ (n factors) by permuting the coordinates and this action defines the semidirect product $G^n \rtimes S_n$ of G^n by S_n. The group $G^n \rtimes S_n$ is called the *wreath product* of G by S_n and is denoted $G \sim S_n$ (our notation follows [**7**]). We set $G_n = G \sim S_n$. The elements of G_n are the set of all $(n+1)$-tuples $(g_1, \ldots, g_n, \pi)$, where $\pi \in S_n$, and $g_i \in G$ for all i. The multiplication rule and inverse of an element in G_n are given by

$$\begin{aligned}(g_1, \ldots, g_n, \pi)(h_1, \ldots, h_n, \tau) &= (g_1 h_{\pi^{-1}(1)}, \ldots, g_n h_{\pi^{-1}(n)}, \pi\tau),\\ (g_1, \ldots, g_n, \pi)^{-1} &= (g^{-1}_{\pi(1)}, \ldots, g^{-1}_{\pi(n)}, \pi^{-1}).\end{aligned}$$

The complex representation theory of G_n is a classical and well studied topic. Among the many sources we mention James and Kerber [**6**], Macdonald [**7**], and the recent book

of Ceccherini-Silberstein, Scarabotti, and Tolli [**3**]. The basic problem can be stated as follows.

Let $\mathcal{P}$ denote the set of all partitions (there is a unique partition of zero with zero parts) and let $\mathcal{P}_n$ denote the set of all partitions of n. For a finite set X, we define

$$\mathcal{P}(X) = \{\mu \mid \mu : X \to \mathcal{P}\}.$$

For $\mu \in \mathcal{P}(X)$, define $\|\mu\| = \sum_{x\in X} |\mu(x)|$, where $|\mu(x)|$ is the sum of the parts of the partition $\mu(x)$ and define

$$\mathcal{P}_n(X) = \{\mu \in \mathcal{P}(X) \mid \|\mu\| = n\}.$$

Let G_* denote the set of conjugacy classes in G. The conjugacy classes of G_n are parametrized by $\mathcal{P}_n(G_*)$ ([**6, 7, 3**]).

Let $\mathcal{Y}$ denote the set of all Young diagrams (there is a unique Young diagram with zero boxes) and $\mathcal{Y}_n$ denote the set of all Young diagrams with n boxes. For a finite set X, we define

$$\mathcal{Y}(X) = \{\mu \mid \mu : X \to \mathcal{Y}\}.$$

For $\mu \in \mathcal{Y}(X)$, define $\|\mu\| = \sum_{x\in X} |\mu(x)|$, where $|\mu(x)|$ is the number of boxes of the Young diagram $\mu(x)$ and define

$$\mathcal{Y}_n(X) = \{\mu \in \mathcal{Y}(X) \mid \|\mu\| = n\}.$$

Denote by $G^\wedge$ the (finite) set of equivalence classes of finite dimensional complex irreducible representations of G. Given $\sigma \in G^\wedge$, we denote by V^σ the corresponding irreducible G-module. Elements of $\mathcal{Y}(G^\wedge)$ are called *Young G-diagrams* and elements of $\mathcal{Y}_n(G^\wedge)$ are called *Young G-diagrams with n boxes*. Given $\mu \in \mathcal{Y}(G^\wedge)$ and $\sigma \in G^\wedge$, we denote by $\mu \downarrow \sigma$ the set of all Young G-diagrams obtained from μ by removing one of the inner corners in the Young diagram $\mu(\sigma)$.

Let $\mu \in \mathcal{Y}$. A *Young tableau of shape* μ is obtained by taking the Young diagram μ and filling its $|\mu|$ boxes (bijectively) with the numbers $1, 2, \ldots, |\mu|$. A Young tableau is said to be *standard* if the numbers in the boxes strictly increase along each row and each column of the Young diagram of μ. Let $\mathrm{tab}(n, \mu)$, where $\mu \in \mathcal{Y}_n$, denote the set of all standard Young tableaux of shape μ and let $\mathrm{tab}(n) = \cup_{\mu\in\mathcal{Y}_n} \mathrm{tab}(n, \mu)$.

Let $\mu \in \mathcal{Y}(G^\wedge)$. A *Young G-tableau of shape* μ is obtained by taking the Young G-diagram μ and filling its $\|\mu\|$ boxes (bijectively) with the numbers $1, 2, \ldots, \|\mu\|$. A Young G-tableau is said to be *standard* if the numbers in the boxes strictly increase along each row and each column of all Young diagrams occuring in μ. Let $\mathrm{tab}_G(n, \mu)$, where $\mu \in \mathcal{Y}_n(G^\wedge)$, denote the set of all standard Young G-tableaux of shape μ and let $\mathrm{tab}_G(n) = \cup_{\mu\in\mathcal{Y}_n(G^\wedge)} \mathrm{tab}_G(n, \mu)$.

Let $T \in \mathrm{tab}_G(n)$ and $i \in \{1, \ldots, n\}$. If i appears in the Young diagram $\mu(\sigma)$, where μ is the shape of T and $\sigma \in G^\wedge$, we write $r_T(i) = \sigma$.

The complex irreducible representations of G_n are parametrized by $\mathcal{Y}_n(G^\wedge)$ and the basic problem of the representation theory of G_n is to explain this correspondence between irreducible representations of G_n and elements of $\mathcal{Y}_n(G^\wedge)$. This is done in [**7**] using symmetric functions and the characteristic map and in [**6, 3**] using Clifford theory and the little group method.

In [**9**] Pushkarev, building on the Vershik-Okounkov approach in the S_n case [**13, 14, 2**], gave a spectral explanation for this correspondence, namely, an internal analysis of the irreducible representations of G_n yields spectral objects parametrizing the irreducible representations and then a bijection is given between these spectral objects and $\mathcal{Y}_n(G^\wedge)$. This approach is inductive in nature and has the following advantages:

(a) The group G_n can be identified with the subgroup

$$\{(g_1,\ldots,g_n,e,\pi) \mid \pi \in S_{n+1} \text{ with } \pi(n+1)=n+1 \text{ and } g_i \in G,\ 1\leq i\leq n\}$$

of G_{n+1} ($e=$ identity element of G) and we have an infinite chain of finite groups

$$G_1 \subseteq G_2 \subseteq \cdots .$$

As a natural byproduct of the theory we get the branching rule from G_{n+1} to G_n: denote the irreducible G_{n+1}-module corresponding to $\mu \in \mathcal{Y}_{n+1}(G^\wedge)$ by V^μ. Then we have G_n-module isomorphisms

$$V^\mu \;\;\cong\;\; \oplus_{\sigma\in G^\wedge} \dim(V^\sigma)\ \left(\oplus_{\lambda\in\mu\downarrow\sigma} V^\lambda\right).$$

(b) Another natural byproduct of the theory yields a parametrization of the bases of irreducible G_n-modules using standard Young G-tableaux and bases of irreducible G^n-modules. More precisely, for $\mu \in \mathcal{Y}_n(G^\wedge)$, we have a canonical direct sum decomposition of V^μ into subspaces, called Gelfand-Tsetlin subspaces,

$$V^\mu \;\;=\;\; \oplus_{T\in \mathrm{tab}_{G(n,\mu)}} V_T,$$

where each V_T is closed under the action of $G^n = G\times\cdots\times G$ (n factors) and, as a G^n-module, is isomorphic to the irreducible G^n-module

$$V^{r_T(1)} \otimes V^{r_T(2)} \otimes \cdots \otimes V^{r_T(n)}.$$

The present work gives a fully detailed exposition of Pushkarev's theory. Our development, based on a multiplicity free chain of subgroups, is slightly different from the original and is along the following lines.

For $g\in G$ and $1\leq i\leq n$ we denote by $g^{(i)}$ the element $(e,\ldots,e,g,e,\ldots,e,1)\in G_n$, where g is in the ith spot, e denotes the identity element of G, and 1 denotes the identity element of S_n. Denote by $G^{(i)}$ the subgroup $\{g^{(i)} | g\in G\}$ of G_n. Note that $G^{(1)},\ldots,G^{(n)}$ commute. We may also think of S_n as the subgroup $\{(e,\ldots,e,\pi) | \pi\in S_n\}$. We write

the element $(e,\ldots,e,\pi)$ as π. We may thus write an element $(g_1,\ldots,g_n,\pi)\in G_n$ as $g_1^{(1)}\ldots g_n^{(n)}\pi=\pi g_1^{(\pi^{-1}(1))}\ldots g_n^{(\pi^{-1}(n))}=\pi g_{\pi(1)}^{(1)}\ldots g_{\pi(n)}^{(n)}$.

For $n\geq 1$, set $H_{n,n}=G_n$ and consider the following chain of subgroups

$$H_{1,n}\subseteq H_{2,n}\subseteq\cdots\subseteq H_{n,n}, \tag{1}$$

where, for $1\leq i\leq n$,

$$H_{i,n}=\{(g_1,\ldots,g_n,\pi)\in G_n\mid \pi(j)=j \text{ for } i+1\leq j\leq n\}.$$

Note that $H_{1,n}$ is isomorphic to G^n. The following are the main steps in the representation theory of G_n.

(i) A direct argument shows that branching from $H_{i,n}$ to $H_{i-1,n}$ is simple, i.e., multiplicity free.

(ii) Consider an irreducible $H_{m,n}$-module V. Since the branching is simple the decomposition of V into irreducible $H_{m-1,n}$-modules is canonical. Each of these modules, in turn, decompose canonically into irreducible $H_{m-2,n}$-modules. Iterating this construction we get a canonical decomposition of V into irreducible $G^n=H_{1,n}$-modules, called the *Gelfand-Tsetlin decomposition (GZ-decomposition)* of V. The irreducible G^n-modules in this decomposition are called the *Gelfand-Tsetlin subspaces (GZ-subspaces)* of V.

(iii) Let $Z_{m,n}$ denote the center of the group algebra $\mathbb{C}[H_{m,n}]$. The *Gelfand-Tsetlin algebra (GZ-algebra)*, denoted $GZ_{m,n}$, is defined to be the (commutative) subalgebra of $\mathbb{C}[H_{m,n}]$ generated by $Z_{1,n}\cup Z_{2,n}\cup\cdots\cup Z_{m,n}$. It is shown that $GZ_{m,n}$ consists of all elements in $\mathbb{C}[H_{m,n}]$ that act by scalars on the GZ-subspaces in every irreducible representation of $H_{m,n}$. It follows that if we have a finite generating set for $GZ_{m,n}$ then the GZ-subspaces are determined by the eigenvalues on this generating set.

(iv) Following Pushkarev, for $i=1,2,\ldots,n$, we define the (generalized) YJM elements $X_1,X_2,\ldots,X_n$ of $\mathbb{C}[H_{n,n}]$:

$$X_i \;=\; \sum_{k=1}^{i-1}\sum_{g\in G}(g^{-1})^{(k)}g^{(i)}(k,i).$$

Note that $X_1=0$. For an algebra A, let $Z[A]$ denote the center of A. It is shown that $GZ_{m,n}=\langle Z[\mathbb{C}[G^n]],X_1,X_2,\ldots,X_m\rangle$.

(v) By a GZ-subspace of G_n we mean a GZ-subspace in some irreducible representation of G_n. Let W be a GZ-subspace of G_n. Then W is an irreducible G^n-module and hence is isomorphic to $V^{\rho_1}\otimes\cdots\otimes V^{\rho_n}$, where $\rho_i\in G^\wedge$, for all i. We call $\rho=(\rho_1,\ldots,\rho_n)$ the *label* of the GZ-subspace W.

It follows from steps (iii) and (iv) above that a GZ-subspace W of G_n is uniquely determined by its label and the eigenvalues of $X_1,\ldots,X_n$ on W. To a GZ-subspace W we associate the tuple

$$\alpha(W)=(\rho,a_1,a_2,\ldots,a_n),$$

where ρ is the label of W and $a_i =$ eigenvalue of X_i on W. We call $\alpha(W)$ the *weight* of the GZ-subspace W. Define

$$\text{spec}_G(n) = \{\alpha(W) \ : \ W \text{ is a GZ-subspace of } G_n\},$$

called the spectrum of G_n.

We have $\dim GZ_{n,n} = |\text{spec}_G(n)|$. There is a natural equivalence relation $\sim$ on $\text{spec}_G(n)$: for $\alpha, \beta \in \text{spec}_G(n)$, $\alpha \sim \beta$ iff the corresponding GZ-subspaces are in same G_n-irreducible. Clearly, we have $|\text{spec}_G(n)/\sim| = |G_n^\wedge|$.

The representation theory of G_n is governed by the spectral object $\text{spec}_G(n)$.

(vi) In the final step we construct a bijection between $\text{spec}_G(n)$ and $\text{tab}_G(n)$ such that tuples in $\text{spec}_G(n)$ related by $\sim$ go to standard Young G-tableaux of the same shape. This step is carried out inductively using an analysis of the following commutation relations that hold in G_n (where $s_i =$ the Coxeter generator $(i, i+1)$):

(a) $X_1, \ldots, X_n$ commute.

(b) $X_i g^{(l)} = g^{(l)} X_i$, $g \in G, 1 \le i, l \le n$.

(c) $s_i g^{(i)} s_i = g^{(i+1)}$, $g \in G, 1 \le i \le n-1$. In particular, $s_i^2 = 1$.

(d) $s_i g^{(l)} = g^{(l)} s_i$, $1 \le i \le n-1, 1 \le l \le n, l \ne i, i+1$.

(e) $s_i X_i s_i + \sum_{g \in G} g^{(i+1)} s_i (g^{-1})^{(i+1)} = X_{i+1}$, $1 \le i \le n-1$.

(f) $s_i X_l = X_l s_i$, $1 \le i \le n-1, 1 \le l \le n, l \ne i, i+1$.

We now give a brief synopsis of the paper. Section 2 collects some preliminaries on wreath products. Section 3 discusses Gelfand-Tsetlin subspaces, Gelfand-Tsetlin decompositions, and Gelfand-Tsetlin algebras for an inductive chain of finite groups with simple branching. In Section 4 we first show that the chain (1) is multiplicity free and then show that the corresponding Gelfand-Tsetlin algebras are generated over $Z[\mathbb{C}[G^n]]$ by the YJM elements, thereby defining the weight of a GZ-subspace and the spectrum $\text{spec}_G(n)$ of G_n. Section 5 describes, using the commutation relations (a)-(f) above, the action of the Coxeter generators on the Gelfand-Tsetlin subspaces in terms of transformations of weights. In Section 6, using the results of Section 5, we give a bijection between $\text{spec}_G(n)$ and $\text{tab}_G(n)$ via the content vectors of standard Young G-tableaux. In Section 7 we study the simplest nontrivial example of the Vershik-Okounkov theory, the classical "Johnson schemes" and the "generalized Johnson schemes". We consider multiplicity free S_n, G_n-actions and explicitly write down the GZ-vectors (in the S_n case) and the GZ-subspaces (in the G_n case) and also identify the irreducibles which occur.

2 Preliminaries

The positive integers are denoted $\mathbb{P}$ and the nonnegative integers are denoted $\mathbb{N}$.

We enumerate the conjugacy classes of G as $G_* = \{C_1, \ldots, C_t\}$ and assume that

$C_1 = \{e\}$. We say that $g \in G$ is of *type* j if $g \in C_j$. Define an involution $\mathcal{I} : \{1, \ldots, t\} \to \{1, \ldots, t\}$ as follows: $\mathcal{I}(j) = j'$ if j' is the type of g^{-1}, for $g \in C_j$.

Let $h = (g_1, \ldots, g_n, \pi) \in G_n$ and let $\tau = (i_1, i_2, \ldots, i_k)$ be a k-cycle in π. The element $g_{i_k} g_{i_{k-1}} \cdots g_{i_1} \in G$ is called the *cycle product* of h corresponding to the cycle τ of π and its type is easily seen to be independent of the order in which the elements of τ are listed. Thus we may define $\rho_h : G_* \to \mathcal{P}$ by

$$\rho_h(C_i) = \text{ Multiset of lengths of all cycles of } \pi \text{ whose cycle product lies in } C_i,\ 1 \leq i \leq t.$$

Clearly $\rho_h \in \mathcal{P}_n(G_*)$. We say that ρ_h is the *type* of $h \in G_n$.

Suppose two elements $(g_1, \ldots, g_n, \pi)$ and $(f_1, \ldots, f_n, \tau)$ are conjugate in G_n. Then we have

$$(h_1, \ldots, h_n, \sigma)(g_1, \ldots, g_n, \pi)(h^{-1}_{\sigma(1)}, \ldots, h^{-1}_{\sigma(n)}, \sigma^{-1}) \quad = \quad (f_1, \ldots, f_n, \tau), \tag{2}$$

for some $h_1, \ldots, h_n \in G$ and $\sigma \in S_n$. Thus $\tau = \sigma\pi\sigma^{-1}$ and τ and π are conjugate in S_n.

We now want to consider the cycle products in $(g_1, \ldots, g_n, \pi)$ and $(f_1, \ldots, f_n, \tau)$. For simplicity we shall write the element $(g_1, \ldots, g_n, \pi)$ as $(\ldots, g_i, \ldots, \pi)$ (it being understood that g_i is in the ith spot). We have

$$\begin{aligned}
&(\ldots, h_i, \ldots, \sigma)(\ldots, g_i, \ldots, \pi)(\ldots, h^{-1}_{\sigma(i)}, \ldots, \sigma^{-1}) \\
&\quad = \ (\ldots, h_i g_{\sigma^{-1}(i)}, \ldots, \sigma\pi)(\ldots, h^{-1}_{\sigma(i)}, \ldots, \sigma^{-1}) \\
&\quad = \ (\ldots, h_i g_{\sigma^{-1}(i)} h^{-1}_{\sigma\pi^{-1}\sigma^{-1}(i)}, \ldots, \sigma\pi\sigma^{-1}) \\
&\quad = \ (\ldots, h_i g_{\sigma^{-1}(i)} h^{-1}_{\tau^{-1}(i)}, \ldots, \tau) \\
&\quad = \ (\ldots, f_i, \ldots, \tau)
\end{aligned}$$

Let $(i_1, \ldots, i_k)$ be a cycle in π. Then $(\sigma(i_1), \ldots, \sigma(i_k))$ is a cycle in τ. We have, using the calculation above,

$$\begin{aligned}
f_{\sigma(i_k)} &= h_{\sigma(i_k)} g_{\sigma^{-1}(\sigma(i_k))} h^{-1}_{\tau^{-1}(\sigma(i_k))} &(3)\\
&= h_{\sigma(i_k)} g_{i_k} h^{-1}_{\sigma(i_{k-1})}. &(4)
\end{aligned}$$

Thus we have (using $\tau^{-1}(\sigma(i_1)) = \sigma(i_k)$)

$$\begin{aligned}
&f_{\sigma(i_k)} f_{\sigma(i_{k-1})} \cdots f_{\sigma(i_1)} \\
&\quad = \ (h_{\sigma(i_k)} g_{i_k} h^{-1}_{\sigma(i_{k-1})})(h_{\sigma(i_{k-1})} g_{i_{k-1}} h^{-1}_{\sigma(i_{k-2})}) \cdots (h_{\sigma(i_1)} g_{i_1} h^{-1}_{\sigma(i_k)}) \\
&\quad = \ h_{\sigma(i_k)} g_{i_k} \cdots g_{i_1} h^{-1}_{\sigma(i_k)}.
\end{aligned}$$

Thus the type of the cycle products $g_{i_k} \cdots g_{i_1}$ and $f_{\sigma(i_k)} \cdots f_{\sigma(i_1)}$ are the same. It follows that if two elements of G_n are conjugate then they have the same type.

Conversely, suppose that $(g_1, \ldots, g_n, \pi), (f_1, \ldots, f_n, \tau) \in G_n$ have the same type. Then we can easily write down a $\sigma \in S_n$ such that $\sigma\pi\sigma^{-1} = \tau$ and such that, for every cycle $(i_1, \ldots, i_k)$ of π, the cycle products $g_{i_k} \cdots g_{i_1}$ and $f_{\sigma(i_k)} \cdots f_{\sigma(i_1)}$ have the same type. Now, using (4), we can find $h_1, \ldots, h_n \in G$ such that (2) holds. It follows that two elements of G_n are conjugate if and only if they have the same type.

An element $g = (g_1, \ldots, g_n, \pi) \in G_n$ is said to be a *nontrivial cycle of type* j if (exactly) one of the following conditions hold:

(i) All cycles of π have length 1 (i.e., π is the identity permutation) and, for some $1 \leq i \leq n$, $g_l = e$ for $l \neq i$, g_i is of type j, and $2 \leq j \leq t$. We say that $\{i\}$ is the *support* of g. We say that g is a *nontrivial* 1-*cycle* of type j.

(ii) There is exactly one cycle, say $(i_1, \ldots, i_k)$, in the cycle decomposition of π of length ≥ 2, the cycle product $g_{i_k} \ldots g_{i_1}$ is of type j, and $g_l = e$, for $l \notin \{i_1, \ldots, i_k\}$. We say that $\{i_1, \ldots, i_k\}$ is the *support* of g. Note that in this case there is no restriction on j, i.e., $1 \leq j \leq t$. We say that g is a *nontrivial* k-*cycle* of type j.

Just as in the S_n case every element of G_n can be written as a product of commuting nontrivial cycles with disjoint support.

By a *nontrivial part* of a partition we mean a part ≥ 2. For a partition $\mu \in \mathcal{P}$ we denote by $\#\mu$ the sum of all the nontrivial parts (with multiplicity) of μ.

Let $\rho \in \mathcal{P}_n(G_*)$. By a *part* of ρ we mean a pair (k, j), where $k \in \mathbb{P}$, $j \in \{1, \ldots, t\}$, and k is a part of $\rho(C_j)$. We may specify ρ by giving its multiset of parts (for example, if k appears m times in $\rho(C_j)$ then the part (k, j) appears m times in the multiset of parts). We say the part (k, j) is *nontrivial* if $(k, j) \neq (1, 1)$. We define

$$\#\rho \;=\; \sum_{j=2}^{t} |\rho(C_j)| + \#(\rho(C_1)),$$

i.e., $\#\rho$ is the sum of the first components (with multiplicity) of all the nontrivial parts of ρ.

For a permutation $s \in S_n$ we denote by $\ell(s)$ the number of inversions in s. It is well known that s can be written as a product of $\ell(s)$ Coxeter transpositions $s_i = (i, i+1)$, $i = 1, 2, \ldots, n-1$ and that s cannot be written as a product of fewer Coxeter transpositions.

All our algebras are finite dimensional, over $\mathbb{C}$, and have units. Subalgebras contain the unit, and algebra homomorphisms preserve units. Given elements or subalgebras $A_1, A_2, \ldots, A_n$ of an algebra A we denote by $\langle A_1, A_2, \ldots, A_n \rangle$ the subalgebra of A generated by $A_1 \cup A_2 \cup \cdots \cup A_n$.

If A is an algebra and $\rho : A \to \mathrm{End}(V)$ is a representation then we use several notations for the action of A on the elements of V. For $a \in A$ and $v \in V$ we set

$$\rho(a)(v) = a \cdot v = av = a(v).$$

Similarly, for $a \in A$ and $W \subseteq V$ we set

$$\rho(a)(W) = a \cdot W = aW = a(W).$$

3 Gelfand-Tsetlin subspaces, Gelfand-Tsetlin decomposition, and Gelfand-Tsetlin algebras

The fundamental building blocks of the spectral approach to the representation theory of S_n and G_n are the concepts of Gelfand-Tsetlin subspaces (GZ-subspaces), Gelfand-Tsetlin decompositions (GZ-decompositions), and Gelfand-Tsetlin algebras (GZ-algebras), together with a convenient set of generators for the GZ algebras, for an inductive chain of finite groups with simple branching. We discuss this in the present and next sections.

Let

$$F_1 \subseteq F_2 \subseteq \cdots \subseteq F_n \tag{5}$$

be an inductive chain of finite groups. Note that we have not assumed that F_1 is the trivial group with one element. We call F_1 the *base group*. Define the following directed graph, called the *branching multigraph or Bratelli diagram* of this chain: its vertices are the elements of the set

$$\coprod_{i=1}^{n} F_i^{\wedge} \quad \text{(disjoint union)}$$

and two vertices μ, λ are joined by k directed edges from μ to λ whenever $\mu \in F_{i-1}^{\wedge}$ and $\lambda \in F_i^{\wedge}$ for some i, and the multiplicity of μ in the restriction of λ to F_{i-1} is k. We say that $F_i^{\wedge}$ is *level* i of the branching multigraph. We write $\mu \nearrow \lambda$ if there is an edge from μ to λ.

For the rest of this section assume that the branching multigraph defined above is actually a graph, i.e., the multiplicities of all restrictions are 0 or 1. We say that the *branching or multiplicities* are *simple*.

Consider the F_n-module V^{λ}, where $\lambda \in F_n^{\wedge}$. Since the branching is simple, the decomposition

$$V^{\lambda} = \bigoplus_{\mu} V^{\mu},$$

where the sum is over all $\mu \in F_{n-1}^{\wedge}$ with $\mu \nearrow \lambda$, is canonical. Iterating this decomposition we obtain a canonical decomposition of V^{λ} into irreducible F_1-modules, i.e.,

$$V^{\lambda} = \bigoplus\nolimits_T V_T, \tag{6}$$

where the sum is over all possible chains

$$T \;=\; \lambda_1 \nearrow \lambda_2 \nearrow \cdots \nearrow \lambda_n, \tag{7}$$

with $\lambda_i \in F_i^\wedge$ and $\lambda_n = \lambda$.

We call (6) the *Gelfand-Tsetlin decomposition (GZ-decomposition)* of V^λ and we call each V_T in (6) a *Gelfand-Tsetlin subspace (GZ-subspace)* of V^λ. By the definition of V_T, we have, for $v_T \in V_T$,

$$\mathbb{C}[F_i] \cdot v_T \quad = \quad V^{\lambda_i}, \quad i = 1, 2, \ldots, n.$$

Also note that chains in (7) are in bijection with directed paths in the branching graph from an element λ_1 of $F_1^\wedge$ to λ.

Fix a *distinguished basis* B^μ for each $V^\mu, \lambda \in F_1^\wedge$. Considering the algebra isomorphism

$$\mathbb{C}[F_n] \quad \cong \quad \bigoplus_{\lambda \in F_n^\wedge} \mathrm{End}(V^\lambda), \tag{8}$$

given by

$$g \mapsto (V^\lambda \xrightarrow{g} V^\lambda \ : \ \lambda \in F_n^\wedge), \quad g \in F_n,$$

we can define three natural subalgebras of $\mathbb{C}[F_n]$ based on the GZ-decomposition (6).

$$\begin{aligned}
\mathcal{A}_0(n) \quad &= \quad \{a \in \mathbb{C}[F_n] : a \text{ acts by a scalar on each GZ-subspace of } V^\lambda, \text{ for all } \lambda \in F_n^\wedge\}, \\
\mathcal{A}_1(n) \quad &= \quad \{a \in \mathbb{C}[F_n] : a \text{ acts diagonally in the distinguished basis } B^\lambda \text{ of each} \\
&\qquad\qquad \text{GZ-subspace of } V^\lambda, \text{ for all } \lambda \in F_n^\wedge\}, \\
\mathcal{A}_2(n) \quad &= \quad \{a \in \mathbb{C}[F_n] : \text{each GZ-subspace of } V^\lambda \text{ is } a \text{ invariant}, \text{ for all } \lambda \in F_n^\wedge\}.
\end{aligned}$$

Clearly, $\mathcal{A}_0(n) \subseteq \mathcal{A}_1(n) \subseteq \mathcal{A}_2(n)$, $\mathcal{A}_0(1) = Z[\mathbb{C}[F_1]]$ and $\mathcal{A}_2(1) = \mathbb{C}[F_1]$.

For each $\lambda \in F_n^\wedge$ and $\mu \in F_1^\wedge$, let $m_{\lambda\mu}$ be the number of GZ-subspaces of V^λ isomorphic to V^μ, i.e., $m_{\lambda\mu}$ is the number of directed paths from μ to λ in the branching graph. It is easily seen that $\mathcal{A}_1(n)$ is a maximal commutative subalgebra of $\mathbb{C}[F_n]$ and that

$$\dim \mathcal{A}_0(n) \quad = \quad \sum_{\lambda \in F_n^\wedge} \sum_{\mu \in F_1^\wedge} m_{\lambda\mu}, \tag{9}$$

$$\dim \mathcal{A}_1(n) \quad = \quad \sum_{\lambda \in F_n^\wedge} \sum_{\mu \in F_1^\wedge} m_{\lambda\mu} \dim V^\mu, \tag{10}$$

$$\dim \mathcal{A}_2(n) \quad = \quad \sum_{\lambda \in F_n^\wedge} \sum_{\mu \in F_1^\wedge} m_{\lambda\mu} \, (\dim V^\mu)^2. \tag{11}$$

We denote $Z[\mathbb{C}[F_i]]$ by Z_i.

Theorem 3.1 *We have*

(i) $\mathcal{A}_0(n) = \langle Z_1, Z_2, \ldots, Z_n \rangle$.

(ii) $\mathcal{A}_1(n) = \langle \mathcal{A}_1(1), Z_1, Z_2, \ldots, Z_n \rangle$.

(iii) $\mathcal{A}_2(n) = \langle \mathbb{C}[F_1], Z_1, Z_2, \ldots, Z_n \rangle$.

Proof (i) Consider the chain T from (7) above. For $i = 1, 2, \ldots, n$, let $p_{\lambda_i} \in Z_i$ denote the primitive central idempotent corresponding to the representation $\lambda_i \in F_i^\wedge$. Define $p_T \in \langle Z_1, Z_2, \ldots, Z_n \rangle$ by

$$p_T = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_n}.$$

A little reflection shows that the image of p_T under the isomorphism (8) is $(f_\mu \ : \ \mu \in F_n^\wedge)$, where $f_\mu = 0$, if $\mu \neq \lambda$ and f_λ is the projection on V_T (with respect to the decomposition (6) of V^λ). The result follows since the primitive central idempotents corresponding to the irreducible representations of a finite group form a basis of the center of the group algebra of the group.

(ii) Note that $\mathbb{C}[F_1]$ commutes with $Z_1, \ldots, Z_n$. The result now follows from part (i) and the isomorphism (8) with $n = 1$.

(iii) Similar to part (ii). $\square$

We call $\mathcal{A}_0(n)$ the *Gelfand-Tsetlin algebra (GZ-algebra)* of the multiplicity free chain of groups (5) and denote it by GZ_n. Following **[9]** we call $\mathcal{A}_2(n)$ the *generalized Gelfand-Tsetlin algebra*. By a *GZ-subspace of F_n* we mean a GZ-subspace of some irreducible representation V^λ of F_n, $\lambda \in F_n^\wedge$. By a *GZ-vector of F_n* we mean a vector in some GZ-subspace of some irreducible representation V^λ of F_n, $\lambda \in F_n^\wedge$. As an immediate consequence of the theorem above we get the following result.

Lemma 3.2 *(i) Let $v \in V^\lambda$, $\lambda \in F_n^\wedge$. If v is an eigenvector (for the action) of every element of GZ_n, then v belongs to some GZ-subspace of V^λ.*

(ii) Let v, u be two GZ-vectors of F_n. If v and u have the same eigenvalues for every element of GZ_n, then v and u belong to the same GZ-subspace of V^λ, for some $\lambda \in F_n^\wedge$.

In Section 4 we define a multiplicity free chain of subgroups of G_n and consider the corresponding GZ-algebras.

4 Simplicity of branching and Young-Jucys-Murphy elements

Let M be a complex finite dimensional semisimple algebra and let N be a semisimple subalgebra. Define the *relative commutant* of this pair to be the subalgebra

$$Z(M, N) \quad = \quad \{m \in M \mid mn = nm \text{ for all } n \in N\},$$

consisting of all elements of M that commute with N.

The following result is well known. We include a proof for completeness.

Theorem 4.1 *Let M be a complex finite dimensional semisimple algebra and let N be a semisimple subalgebra. Then $Z(M, N)$ is semisimple and the following conditions are equivalent:*

1. The restriction of any finite dimensional complex irreducible representation of M to N is multiplicity free.

2. The relative commutant $Z(M,N)$ is commutative.

Proof By Wedderburn's theorem we may assume, without loss of generality, that $M = M_1 \oplus \cdots \oplus M_k$, where each M_i is a matrix algebra. We write elements of M as $(m_1, \ldots, m_k)$, where $m_i \in M_i$. For $i = 1, \ldots, k$, let N_i denote the image of N under the natural projection of M onto M_i. Being the homomorphic image of a semisimple algebra, N_i itself is semisimple.

We have $Z(M,N) = Z(M_1, N_1) \oplus \cdots \oplus Z(M_k, N_k)$. By the double centralizer theorem each $Z(M_i, N_i)$, and thus $Z(M,N)$, is semisimple.

For $i = 1, \ldots, k$, let V_i denote the M-submodule consisting of all $(m_1, \ldots, m_k) \in M$ with $m_j = 0$ for $j \neq i$ and with all entries of m_i not in the first column equal to zero. Note that $V_1, \ldots, V_k$ are all the distinct inequivalent irreducible M-modules and that the decomposition of V_i into irreducible N-modules is identical to the decomposition of V_i into irreducible N_i-modules.

It now follows from the double centralizer theorem that V_i is multiplicity free as a N_i-module, for all i if and only if all irreducible modules of $Z(M_i, N_i)$ have dimension 1, for all i if and only if $Z(M_i, N_i)$ is abelian, for all i if and only if $Z(M,N)$ is abelian. $\square$

Define the following subalgebras of $\mathbb{C}[H_{n,n}]$:

(i) For $2 \leq m \leq n$, set $Z_{m,m-1,n} = Z[\mathbb{C}[H_{m,n}], \mathbb{C}[H_{m-1,n}]]$.

(ii) For $1 \leq m \leq n$, set $Z_{m,n} = Z[\mathbb{C}[H_{m,n}]]$.

In this section we show that the branching from $H_{m,n}$ to $H_{m-1,n}$, $2 \leq m \leq n$ is multiplicity free and find a convenient set of generators, the Young-Jucys-Murphy elements, of the Gelfand-Tsetlin algebra of the chain of groups,

$$H_{1,n} \subseteq H_{2,n} \subseteq \cdots \subseteq H_{n,n}, \tag{12}$$

over the center $Z_{1,n}$ of the group algebra of the base group $H_{1,n} = G^n$.

For $i = 1, \ldots, n-1$ and $j = 1, \ldots, t$ define the following elements of $\mathbb{C}[H_{n,n}]$:

$$Y_{i,j} = \sum_{\tau} \tau,$$

where the sum is over all elements $\tau \in H_{n,n}$ satisfying the following properties: τ is a nontrivial i-cycle of type j and n does not belong to the nontrivial cycle. Note that $Y_{1,1} = 0$ and that all other $Y_{i,j}$ are nonzero. We also set $Y_{n,j} = 0$ for all $1 \leq j \leq t$.

For $i = 1, \ldots, n$ and $j = 1, \ldots, t$ define the following elements of $\mathbb{C}[H_{n,n}]$:

$$Y'_{i,j} = \sum_{\tau} \tau,$$

where the sum is over all elements $\tau \in H_{n,n}$ satisfying the following properties: τ is a nontrivial i-cycle of type j and n belongs to the nontrivial cycle. Note that $Y'_{1,1} = 0$ and that all other $Y'_{i,j}$ are nonzero.

Define

$$\mathcal{P}'_n(G_*) \quad = \quad \{(\rho, \lambda, j) |\, \rho \in \mathcal{P}_n(G_*), \lambda \in \mathbb{P}, j \in \{1, \ldots, t\} \text{ with } \lambda \in \rho(C_j)\}.$$

For $(\rho, \lambda, j) \in \mathcal{P}'_n(G_*)$ define $c_{(\rho,\lambda,j)} \in \mathbb{C}[H_{n,n}]$ to be the sum of all elements $\tau \in H_{n,n}$ satisfying

(i) type$(\tau) = \rho$.

(ii) size of the cycle of τ containing n is λ and the corresponding cycle product is of type j.

Note that, for $1 \leq i \leq n,\ 1 \leq j \leq t$, $Y_{i,j}$ and $Y'_{i,j}$ are equal to $c_{(\rho,\lambda,j)}$, for suitable choice of ρ, λ, and j.

Lemma 4.2 *(i)* $\{c_{(\rho,\lambda,j)} \mid (\rho, \lambda, j) \in \mathcal{P}'_n(G_*)\}$ *is a basis of* $Z_{n,n-1,n}$. *It follows that*

$$\langle Y_{i,j}, Y'_{i,j} \mid 1 \leq i \leq n,\ 1 \leq j \leq t\rangle \subseteq Z_{n,n-1,n}.$$

(ii) For $(\rho, \lambda, j) \in \mathcal{P}'_n(G_*)$ *we have*

$$c_{(\rho,\lambda,j)} \in \langle Y_{i,p}, Y'_{i,p} \mid 1 \leq p \leq t,\ 1 \leq i \leq k\rangle,$$

where $k = \#\rho$.

(iii) $Z_{n,n-1,n} = \langle Y_{i,j}, Y'_{i,j} \mid 1 \leq j \leq t,\ 1 \leq i \leq n\rangle$.

(iv) $Z_{n-1,n} = \langle Y_{i,j}, Y'_{1,j} \mid 1 \leq j \leq t,\ 1 \leq i \leq n-1\rangle$.

Proof (i) Let $\tau, \tau' \in H_{n,n}$. Then, using the same argument that characterized conjugacy in G_n in Section 2, we can show that $\tau = \sigma\tau'\sigma^{-1}$ for some $\sigma \in H_{n-1,n}$ iff type$(\tau) =$ type(τ'), the length of the cycle containing n is same in both τ and τ', and the cycle products of the cycles containing n are of the same type in both τ and τ'. The result follows.

(ii) By induction on $\#\rho$. If $\#\rho = 0$, then $c_{(\rho,\lambda,j)}$ is the identity element of $\mathbb{C}[H_{n,n}]$ and the result is clearly true. Assume the result whenever $\#\rho \leq l$. Consider $(\rho, \lambda, j) \in \mathcal{P}'_n(G_*)$ with $\#\rho = l + 1$.

Denote the multiset of nontrivial parts of ρ by $\{(\lambda_1, j_1), (\lambda_2, j_2), \ldots, (\lambda_m, j_m)\}$. Consider the following two subcases:

(a) $\lambda = 1$ and $j = 1$: Consider the product $Y_{\lambda_1,j_1}Y_{\lambda_2,j_2}\cdots Y_{\lambda_m,j_m}$. Using part (i) we see that this product is in $Z_{n,n-1,n}$ and thus can be expanded in the basis given in part (i). A little reflection shows that

$$Y_{\lambda_1,j_1}Y_{\lambda_2,j_2}\cdots Y_{\lambda_m,j_m} \quad = \quad \alpha_{(\rho,\lambda,j)}c_{(\rho,\lambda,j)} + \sum_{(\rho',\lambda',j')} \alpha_{(\rho',\lambda',j')}c_{(\rho',\lambda',j')},$$

where $\alpha_{(\rho,\lambda,j)} \in \mathbb{P}$, $\alpha_{(\rho',\lambda',j')} \in \mathbb{N}$ and the sum is over all (ρ', λ', j') with $\#\rho' < \#\rho$. The result follows by induction.

(b) $\lambda \neq 1$ or $j \neq 1$: Without loss of generality we may assume $(\lambda, j) = (\lambda_1, j_1)$. Now consider the product

$$Y'_{\lambda_1,j_1} Y_{\lambda_2,j_2} \cdots Y_{\lambda_m,j_m} \quad = \quad \alpha_{(\rho,\lambda,j)} c_{(\rho,\lambda,j)} + \sum_{(\rho',\lambda',j')} \alpha_{(\rho',\lambda',j')} c_{(\rho',\lambda',j')},$$

where $\alpha_{(\rho,\lambda,j)} \in \mathbb{P}$, $\alpha_{(\rho',\lambda',j')} \in \mathbb{N}$ and the sum is over all (ρ', λ', j') with $\#\rho' < \#\rho$. The result follows by induction.

(iii) Follows from parts (i) and (ii).

(iv) Embed $H_{n-1,n-1}$ into $H_{n-1,n}$ in the obvious way giving rise to an embedding $\phi : Z_{n-1,n-1} \to Z_{n-1,n}$. Note that $Z_{n-1,n}$ is isomorphic to the tensor product of $\phi(Z_{n-1,n-1})$ and $Z[\mathbb{C}[G^{(n)}]]$. Now $Z[\mathbb{C}[G^{(n)}]]$ is generated by $Y'_{1,j}, 1 \leq j \leq t$ and a proof similar to the proof of part (iii) shows that $\phi(Z_{n-1,n-1})$ is generated by $Y_{i,j}, 1 \leq j \leq t, 1 \leq i \leq n-1$. The result follows. □

For $i = 1, \ldots, n$ define the following elements of $\mathbb{C}[H_{n,n}]$:

$$X_i \quad = \quad \sum_{k=1}^{i-1} \sum_{g \in G} (g^{-1})^{(k)} g^{(i)}(k, i).$$

Note that $X_1 = 0$. It is easy to see that X_i is the difference of an element in $Z_{i,n}$ and an element in $Z_{i-1,n}$. These elements are called the *Young-Jucys-Murphy* (YJM) elements.

Theorem 4.3 *(i)* $Z_{m,m-1,n} = \langle Z_{m-1,n}, X_m \rangle,\ 2 \leq m \leq n$.

(ii) $Z_{m,m-1,n},\ 2 \leq m \leq n$ *is commutative.*

Proof (i) We first consider $m = n$. Clearly $Z_{n,n-1,n} \supseteq \langle Z_{n-1,n}, X_n \rangle$ (note that $X_n = Y'_{2,1}$). To show the converse we need to show (by Lemma 4.2 (iii) and (iv)) that $Y'_{i,j} \in \langle Z_{n-1,n}, X_n \rangle$, for $i = 2, \ldots, n$ and $j = 1, \ldots, t$.

Observe that, for $2 \leq i \leq n$ and $2 \leq j \leq t$, we have

$$Y'_{i,j} \quad = \quad Y'_{1,j} Y'_{i,1}. \tag{13}$$

Since $Y'_{1,j} \in Z_{n-1,n}$, for $1 \leq j \leq t$ it is enough to show that $Y'_{i,1} \in \langle Z_{n-1,n}, X_n \rangle$, for $i = 2, \ldots, n$. We show this by induction on i.

Since $Y'_{2,1} = X_n$ we have $Y'_{2,1} \in \langle Z_{n-1,n}, X_n \rangle$. Suppose $Y'_{2,1}, \ldots, Y'_{k+1,1} \in \langle Z_{n-1,n}, X_n \rangle$. We shall now show that $Y'_{k+2,1} \in \langle Z_{n-1,n}, X_n \rangle$.

We write $Y'_{k+1,1}$ as

$$\sum_{i_1,\ldots,i_k,g_1,\ldots,g_{k+1}} g_1^{(i_1)} g_2^{(i_2)} \cdots g_k^{(i_k)} g_{k+1}^{(n)}(i_1, \ldots, i_k, n),$$

where the sum is over all $(i_1,\ldots,i_k)\in\{1,\ldots,n-1\}^k$ with distinct components and all $(g_1,\ldots,g_{k+1})\in G^{k+1}$ with $g_{k+1}g_k\cdots g_2g_1=e$. In the following we use this summation convention implicitly.

Now consider the product $Y'_{k+1,1}X_n\in\langle Z_{n-1,n},X_n\rangle$:

$$\left\{\sum_{i_1,\ldots,i_k,g_1,\ldots,g_{k+1}} g_1^{(i_1)}g_2^{(i_2)}\cdots g_k^{(i_k)}g_{k+1}^{(n)}(i_1,\ldots,i_k,n)\right\}\left\{\sum_{i=1}^{n-1}\sum_g (g^{-1})^{(i)}g^{(n)}(i,n)\right\}. \quad (14)$$

Take a typical element

$$g_1^{(i_1)}g_2^{(i_2)}\cdots g_k^{(i_k)}g_{k+1}^{(n)}(i_1,\ldots,i_k,n)(g^{-1})^{(i)}g^{(n)}(i,n)$$

of this product. If $i\neq i_l$, for $l=1,\ldots,k$, this product is

$$(g^{-1})^{(i)}(g_1g)^{(i_1)}g_2^{(i_2)}\cdots g_k^{(i_k)}g_{k+1}^{(n)}(i,i_1,\ldots,i_k,n).$$

Note that $g_{k+1}\cdots g_2(g_1g)(g^{-1})=e$.

On the other hand if $i=i_l$, for some $1\le l\le k$, this product becomes

$$(g_1g)^{(i_1)}g_2^{(i_2)}\cdots g_l^{(i_l)}(g_{l+1}g^{-1})^{(i_{l+1})}g_{l+2}^{(i_{l+2})}\cdots g_{k+1}^{(n)}(i_1,\ldots,i_l)(i_{l+1},\ldots,n).$$

Note that, since $g_{k+1}g_k\cdots g_1=e$, we have

$$g_{k+1}\cdots g_{l+2}(g_{l+1}g^{-1})=g(g_l\cdots g_2(g_1g))^{-1}g^{-1}.$$

It follows that the element (14) above is equal to

$$\sum_{i,i_1,\ldots,i_k,g,g_1,\ldots,g_{k+1}} g^{(i)}g_1^{(i_1)}\cdots g_k^{(i_k)}g_{k+1}^{(n)}(i,i_1,\ldots,i_k,n) \quad (15)$$
$$+\sum_{i_1,\ldots,i_k}\sum_{l=1}^{k}\sum_{j=1}^{t}\sum_{g_1,\ldots,g_{k+1}}\frac{|G|}{|C_j|}g_1^{(i_1)}\cdots g_k^{(i_k)}g_{k+1}^{(n)}(i_1,\ldots,i_l)(i_{l+1},\ldots,i_k,n),$$

where the first sum is over all $(i,i_1,\ldots,i_k)\in\{1,2,\ldots,n-1\}^{k+1}$ with distinct components and all $(g,g_1,\ldots,g_{k+1})\in G^{k+2}$ with $g_{k+1}\cdots g_1g=e$ and the second sum is over all $(i_1,\ldots,i_k)\in\{1,2,\ldots,n-1\}^k$ with distinct components and all $(g_1,\ldots,g_{k+1})\in G^{k+1}$ with type of $g_{k+1}\cdots g_{l+1}$ equal to j and the type of $g_l\cdots g_1$ equal to $\mathcal{I}(j)$.

We can rewrite (15) as

$$Y'_{k+2,1}+\sum_{(\rho,\lambda,j)}\alpha_{(\rho,\lambda,j)}c_{(\rho,\lambda,j)},$$

where $\alpha_{(\rho,\lambda,j)}\in\mathbb{N}$ and the sum is over all (ρ,λ,j) with $\#\rho\le k+1$. By induction hypothesis, (13), and part (ii) of Lemma 4.2 it follows that $Y'_{k+2,1}\in\langle Z_{n-1,n},X_n\rangle$.

We have now shown $Z_{n,n-1,n} = \langle Z_{n-1,n}, X_n \rangle$. The case of general $Z_{m,m-1,n}$ can be shown by embedding $Z_{m,m-1,m}$ in $Z_{m,m-1,n}$ (as in part (iv) of Lemma 4.2).

(ii) This follows from part (i) since $X_m \in Z_{m,n} - Z_{m-1,n}$. $\square$

It follows from Theorem 4.1 and part (ii) of Theorem 4.3 that the chain (12) is multiplicity free. Set, for $1 \leq m \leq n$,

$$GZ_{m,n} = \langle Z_{1,n}, Z_{2,n}, \ldots, Z_{m,n} \rangle,$$

so that $GZ_{n,n}$ is the Gelfand-Tsetlin algebra of the chain (12). Note that $X_i \in GZ_{i,n} \subseteq GZ_{n,n}$.

Theorem 4.4 *We have*

$$GZ_{m,n} = \langle Z[\mathbb{C}[G^n]], X_1, X_2, \ldots, X_m \rangle, \ 1 \leq m \leq n.$$

Proof The proof is by induction on n and, for each n, by induction on m. The cases $n = 1, 2$ are clear. Now consider general n. The case $m = 1$ is obvious. Assume we have proved that $GZ_{m-1,n} = \langle Z[\mathbb{C}[G^n]], X_1, X_2, \ldots, X_{m-1} \rangle$. It remains to show that $GZ_{m,n} = \langle GZ_{m-1,n}, X_m \rangle$. The left hand side clearly contains the right hand side so it suffices to check that the left hand side is contained in the right hand side. For this it suffices to check that $Z_{m,n} \subseteq \langle GZ_{m-1,n}, X_m \rangle$. This follows from part (i) of Theorem 4.3 since $Z_{m,n} \subseteq Z_{m,m-1,n}$. $\square$

Let V be a GZ-subspace of G_n. Then V is an irreducible G^n-module and is thus isomorphic to $\rho_1 \otimes \cdots \otimes \rho_n, \ \rho_i \in G^\wedge$ for all i. We call $\rho = (\rho_1, \ldots, \rho_n)$ the *label* of V.

Define

$$\alpha(V) \ = \ (\rho, \alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n,$$

where $\alpha_i =$ eigenvalue of X_i on V. We call $\alpha(v)$ the *weight* of V (note that $\alpha_1 = 0$ since $X_1 = 0$). Define the *spectrum* of G_n by

$$\mathrm{spec}_G(n) = \{\alpha(V) \ : \ V \text{ is a GZ-subspace of } G_n\}.$$

Let V be a GZ-subspace of G_n with label ρ. Then the primitive central idempotent in $Z[\mathbb{C}[G^n]]$ corresponding to ρ will have eigenvalue 1 on V and eigenvalue 0 on GZ-subspaces with different labels. It now follows from Lemma 3.2 and Theorem 4.4 that a GZ-subspace is uniquely determined by its weight.

By definition of GZ-subspaces and Lemma 3.2, the set $\mathrm{spec}_G(n)$ is in natural bijection with chains

$$T \ = \ \lambda_1 \nearrow \lambda_2 \nearrow \cdots \nearrow \lambda_n, \tag{16}$$

where $\lambda_i \in H^\wedge_{i,n}, \ 1 \leq i \leq n$, in the Bratelli diagram of (12).

Given $\alpha \in \text{spec}_G(n)$ we denote by V_α the GZ-subspace with weight α and by T_α the corresponding chain in the branching graph. Similarly, given a chain T as in (16) we denote the correponding GZ-subspace by V_T and the weight vector $\alpha(V_T)$ by $\alpha(T)$. Thus we have 1-1 correspondences

$$T \mapsto \alpha(V_T), \quad \alpha \mapsto T_\alpha$$

between chains (16) and $\text{spec}_G(n)$. For $\lambda \in H^\wedge_{n,n}$ define

$$\text{spec}_G(n,\lambda) = \{\alpha \in \text{spec}_G(n) | T_\alpha \text{ ends at } \lambda\}.$$

We have, from (9),

$$\begin{aligned} \dim GZ_{n,n} &= |\text{spec}_G(n)|, \\ \dim V^\lambda &= \sum_{\alpha \in \text{spec}_G(n,\lambda)} \dim V_\alpha, \ \lambda \in H^\wedge_{n,n}. \end{aligned}$$

There is a natural equivalence relation $\sim$ on $\text{spec}_G(n)$: for $\alpha, \beta \in \text{spec}_G(n)$,

$$\alpha \sim \beta \quad \Leftrightarrow \quad \alpha, \beta \in \text{spec}_G(n,\lambda) \text{ for some } \lambda \in H^\wedge_{n,n}.$$

Clearly we have $|\text{spec}_G(n)/\sim| = |H^\wedge_{n,n}|$.

5 Action of Coxeter generators on GZ-subspaces

In this section we describe the action of the Coxeter generators on GZ-subspaces in terms of transformations of weights.

Let $\lambda \in H^\wedge_{n,n}$. We have the GZ-decomposition

$$V^\lambda \quad = \quad \oplus_{\alpha \in \text{spec}_G(n,\lambda)} V_\alpha, \tag{17}$$

of V^λ into irreducible G^n-modules.

We now consider the action of the Coxeter generators $s_i = (i, i+1)$ of S_n on V^λ. Since the V_α consist of common eigenvectors of $X_1, \ldots, X_n$ and are G^n-invariant, it is useful to know the commutation relations satisfied by the s_i, the X_j, and the $g^{(l)}$.

Lemma 5.1 *The following relations hold in G_n:*

(i) $X_1, \ldots, X_n$ *commute.*

(ii) $X_i g^{(l)} = g^{(l)} X_i,\ g \in G,\ 1 \le i, l \le n.$

(iii) $s_i g^{(i)} s_i = g^{(i+1)},\ g \in G,\ 1 \le i \le n-1.$ *In particular,* $s_i^2 = 1,\ 1 \le i \le n-1.$

(iv) $s_i g^{(l)} = g^{(l)} s_i,\ 1 \le i \le n-1,\ 1 \le l \le n,\ l \ne i, i+1.$

(v) $s_i X_i s_i + \sum_{g \in G} g^{(i+1)} s_i (g^{-1})^{(i+1)} = X_{i+1},\ 1 \le i \le n-1.$

(vi) $s_i X_l = X_l s_i,\ 1 \le i \le n-1,\ 1 \le l \le n,\ l \ne i, i+1.$

Proof (i) We have already seen this.

(ii) This can be checked directly. An alternative proof is as follows. On every GZ-subspace of G_n, the actions of X_i and $g^{(l)}$ clearly commute. By considering the isomorphism

$$\mathbb{C}[G_n] \;\cong\; \bigoplus_{\lambda \in G_n^\wedge} \mathrm{End}(V^\lambda),$$

given by

$$g \mapsto (V^\lambda \xrightarrow{g} V^\lambda \;:\; \lambda \in G_n^\wedge), \quad g \in G_n,$$

we see that X_i and $g^{(l)}$ commute in G_n.

(iii) and (iv) This is clear.

(v) We have

$$\begin{aligned} s_i X_i s_i &= (i,i+1)\left(\sum_{k=1}^{i-1}\sum_{g\in G}(g^{-1})^{(k)} g^{(i)}(k,i)\right)(i,i+1) \\ &= \sum_{k=1}^{i-1}\sum_{g\in G}(g^{-1})^{(k)} g^{(i+1)}(k,i+1) \\ &= X_{i+1} - \sum_{g\in G}(g^{-1})^{(i)} g^{(i+1)}(i,i+1). \end{aligned}$$

(vi) First assume $l \leq i-1$. Then

$$\begin{aligned} s_i X_l &= (i,i+1)\left(\sum_{k=1}^{l-1}\sum_{g\in G}(g^{-1})^{(k)} g^{(l)}(k,l)\right) \\ &= \left(\sum_{k=1}^{l-1}\sum_{g\in G}(g^{-1})^{(k)} g^{(l)}(k,l)\right)(i,i+1) \\ &= X_l s_i. \end{aligned}$$

Now assume $l \geq i+2$. Then

$$\begin{aligned} s_i X_l &= (i,i+1)\left(\sum_{k=1}^{i-1}\sum_{g\in G}(g^{-1})^{(k)}(g)^{(l)}(k,l) + \sum_{k=i+2}^{l-1}\sum_{g\in G}(g^{-1})^{(k)}(g)^{(l)}(k,l)\right. \\ &\quad \left. + \sum_{g\in G}(g^{-1})^{(i)}(g)^{(l)}(i,l) + \sum_{g\in G}(g^{-1})^{(i+1)}(g)^{(l)}(i+1,l)\right) \\ &= \left(\sum_{k=1}^{i-1}\sum_{g\in G}(g^{-1})^{(k)}(g)^{(l)}(k,l) + \sum_{k=i+2}^{l-1}\sum_{g\in G}(g^{-1})^{(k)}(g)^{(l)}(k,l)\right. \\ &\quad \left. + \sum_{g\in G}(g^{-1})^{(i+1)}(g)^{(l)}(i+1,l) + \sum_{g\in G}(g^{-1})^{(i)}(g)^{(l)}(i,l)\right)(i,i+1) \\ &= X_l s_i. \;\square \end{aligned}$$

Using part (iii) of Lemma 5.1 we can rewrite part (v) of Lemma 5.1 as

$$X_i s_i + \sum_{g\in G} g^{(i)}(g^{-1})^{(i+1)} \quad = \quad s_i X_{i+1},\ 1 \le i \le n-1. \tag{18}$$

Consider the irreducible G^n-module V_α in the decomposition (17) above. Let V be the subspace of V^λ spanned by V_α and $s_i \cdot V_\alpha$. Lemma 5.1 shows that V is invariant under the actions of s_i, X_i, X_{i+1}, and G^n. A study of this action will enable us to write down matrices for the action of s_i on V_α.

Lemma 5.2 *For $i = 1, 2, \ldots, n-1$, let A_i be the subalgebra of $\mathbb{C}[G_n]$ generated by G^n, s_i, X_i, and X_{i+1}. Then A_i is semisimple and the actions of X_i and X_{i+1} on any A_i-module are simultaneously diagonalizable.*

Proof Let $Mat(n)$ denote the algebra of complex $(|G|^n n!) \times (|G|^n n!)$ matrices, with rows and columns indexed by elements of G_n. Consider the left regular representation of G_n. Writing this in matrix terms gives an embedding of $\mathbb{C}[G_n]$ into $Mat(n)$. We write $\gamma : \mathbb{C}[G_n] \hookrightarrow Mat(n)$.

Note that

(a) The left action of $(i, i+1)$ on G_n is inverse to itself.

(b) For $k < i$, the left action of $(g^{-1})^{(k)} g^{(i)}(k, i)$ on G_n is inverse to itself.

(c) For $g \in G$ and $1 \le l \le n$, the left action of $g^{(l)}$ on G_n is inverse to the action of $(g^{-1})^{(l)}$.

It follows that the matrices $\gamma(s_i), \gamma(X_i), \gamma(X_{i+1})$ are real and symmetric and that the generating set

$$\{\gamma(s_i), \gamma(X_i), \gamma(X_{i+1})\} \cup \{\gamma(g^{(l)}) : 1 \le l \le n, g \in G\}$$

of $\gamma(A_i)$ is closed under the matrix $*$ operation $M \mapsto (\bar{M})^t$. So $\gamma(A_i)$ itself is closed under the $*$ operation and a standard result on finite dimensional C^*-algebras now shows that $\gamma(A_i)$ (and hence A_i) is semisimple.

Part (ii) follows since $\gamma(X_i)$ and $\gamma(X_{i+1})$ are commuting real, symmetric matrices and thus the $*$-subalgebra of $\gamma(A_i)$ generated by them is commutative. $\Box$

Before proceeding further we introduce some useful notation. For $1 \le i \le n-1$, let ω_i be the involution on $\{1, 2, \ldots, n\}$ defined by $\omega_i(l) = l$, if $l \ne i, i+1$, $\omega_i(i) = i+1$, and $\omega_i(i+1) = i$. Parts (iii) and (iv) of Lemma 5.1 may be written as follows. For $1 \le i \le n-1$ and $1 \le l \le n$ we have

$$g^{(l)} s_i \quad = \quad s_i g^{\omega_i(l)}. \tag{19}$$

Let W_1 and W_2 be vector spaces and set $W = W_1 \otimes W_2$. We can define a switch operator on W that sends $w_1 \otimes w_2$ to $w_2 \otimes w_1$. Now let U be a vector space having

the same dimension as W. We can fix an isomorphism between U and W and transfer the switch operator on W to U via this isomorphism. However, since the isomorphism between U and W is not canonical, there is no canonically defined switch operator on U. This situation does not arise when we consider irreducible G^n-modules.

Let $\rho = (\rho_1, \ldots, \rho_n)$, where $\rho_i \in G^\wedge$ for all i and consider the irreducible G^n-module $V^{\rho_1} \otimes \cdots \otimes V^{\rho_n}$. Let $1 \leq i \leq n-1$. Define the involution, called the *switch operator*,

$$\tau_{i,\rho} : V^{\rho_1} \otimes \cdots \otimes V^{\rho_n} \to V^{\rho_1} \otimes \cdots \otimes V^{\rho_{i-1}} \otimes V^{\rho_{i+1}} \otimes V^{\rho_i} \otimes V^{\rho_{i+2}} \otimes \cdots \otimes V^{\rho_n}$$

by switching the i and $i+1$ factors:

$$\tau_{i,\rho}(v_1 \otimes \cdots \otimes v_n) \;=\; v_1 \otimes \cdots \otimes v_{i-1} \otimes v_{i+1} \otimes v_i \otimes v_{i+2} \otimes \cdots \otimes v_n.$$

We have, for $g \in G,\ v \in V^{\rho_1} \otimes \cdots \otimes V^{\rho_n},\ 1 \leq l \leq n$,

$$\tau_{i,\rho}(g^{(l)}v) \;=\; g^{(\omega_i(l))}\tau_{i,\rho}(v). \tag{20}$$

Now let V be an irreducible G^n-module isomorphic to $V^{\rho_1} \otimes \cdots \otimes V^{\rho_n}$. Fix a G^n-linear isomorphism $f : V \to V^{\rho_1} \otimes \cdots \otimes V^{\rho_n}$. Define an involution

$$\tau_{i,V} : V \to V$$

by $\tau_{i,V} = f^{-1}\tau_{i,\rho}f$. It is easily seen by Schur's lemma that $\tau_{i,V}$ is independent of the chosen f and therefore $\tau_{i,V}$ is canonically defined. We have, for $g \in G,\ v \in V,\ 1 \leq l \leq n$,

$$\tau_{i,V}(g^{(l)}v) \;=\; g^{(\omega_i(l))}\tau_{i,V}(v). \tag{21}$$

In what follows we shall use (19), (20), (21) (and (23), (24), (25), (26), (27) below) without explicit mention.

Let V_α be a GZ-subspace of $V^\lambda, \lambda \in H^\wedge_{n,n}$ with weight $\alpha = (\rho, \alpha_1, \ldots, \alpha_n)$, where $\rho = (\rho_1, \ldots, \rho_n)$. Fix a G^n-linear isomorphism

$$f : V_\alpha \to V^{\rho_1} \otimes \cdots \otimes V^{\rho_n}.$$

Let $1 \leq i \leq n-1$. Since $s_i^2 = 1$ the map $v \mapsto s_i \cdot v$ on V^λ is an involution. Consider the subspace s_iV_α of V^λ. Then, by Lemma 5.1 (iii) and (iv), s_iV_α is also closed under the G^n-action. The map

$$f^{\tau_i} : s_iV_\alpha \to V^{\rho_1} \otimes \cdots \otimes V^{\rho_{i-1}} \otimes V^{\rho_{i+1}} \otimes V^{\rho_i} \otimes V^{\rho_{i+2}} \otimes \cdots \otimes V^{\rho_n}, \tag{22}$$

given by $f^{\tau_i}(s_iv) = \tau_{i,\rho}(f(v))$ is a G^n-linear isomorphism. To see this, let $v \in V_\alpha$. Then, for $1 \leq l \leq n$, we have

$$\begin{aligned} g^{(l)}f^{\tau_i}(s_iv) &= g^{(l)}(\tau_{i,\rho}(f(v))), \\ f^{\tau_i}(g^{(l)}s_iv) &= f^{\tau_i}(s_ig^{(\omega_i(l))}v)) \\ &= \tau_{i,\rho}(f(g^{(\omega_i(l))}v)) \\ &= \tau_{i,\rho}(g^{(\omega_i(l))}(f(v))) \\ &= g^{(l)}(\tau_{i,\rho}(f(v))). \end{aligned}$$

For $1 \le i \le n-1$, define an element $b_i = \sum_{g\in G} g^{(i)}(g^{-1})^{(i+1)} \in \mathbb{C}[G_n]$. For $h \in G$ we have $h^{(l)}b_i = b_i h^{(l)},\ l \ne i, i+1$ and

$$h^{(i)}b_i = \textstyle\sum_{g\in G}(hg)^{(i)}(g^{-1})^{(i+1)} = \sum_{g\in G}(hg)^{(i)}((hg)^{-1})^{(i+1)}h^{(i+1)} = b_i h^{(i+1)}.$$

Similarly, we can show $h^{(i+1)}b_i = b_i h^{(i)}$. We have, for $1 \le l \le n$ and $h \in G$,

$$h^{(l)}b_i \;=\; b_i h^{(\omega_i(l))}. \tag{23}$$

Also note that we can rewrite (18) as follows

$$X_i s_i \;=\; s_i X_{i+1} - b_i,\ \ X_{i+1}s_i = s_i X_i + b_i. \tag{24}$$

The map $V_\alpha \to s_i V_\alpha$ given by $v \mapsto s_i b_i v$ is a G^n-linear map. This follows from, for $1 \le l \le n$,

$$s_i b_i(g^{(l)}v) \;=\; s_i(g^{(\omega_i(l))}b_i v) \;=\; g^{(l)}s_i b_i v. \tag{25}$$

It follows that

$$\rho_i \ne \rho_{i+1} \quad \text{implies} \quad b_i v = 0,\ v \in V_\alpha. \tag{26}$$

Now assume $\rho_i = \rho_{i+1}$. The map $V_\alpha \to s_i V_\alpha$ given by $v \mapsto s_i \tau_{i,V_\alpha}(v)$ is a G^n-linear isomorphism. This follows from, for $1 \le l \le n$,

$$s_i\tau_{i,V_\alpha}(g^{(l)}v) \;=\; s_i(g^{(\omega_i(l))}\tau_{i,V_\alpha}(v)) \;=\; g^{(l)}s_i\tau_{i,V_\alpha}(v).$$

It follows that $b_i v = \beta\tau_{i,V_\alpha}(v), v \in V_\alpha$, for some scalar β. Now the trace of the action of b_i on $V^{\rho_1} \otimes \cdots \otimes V^{\rho_n}$ is

$$\frac{\dim(V^{\rho_1})\cdots\dim(V^{\rho_n})}{\dim(V^{\rho_i})\dim(V^{\rho_{i+1}})}\sum_{g\in G}\chi(g)\chi(g^{-1}) = \frac{\dim(V^{\rho_1})\cdots\dim(V^{\rho_n})}{\dim(V^{\rho_i})\dim(V^{\rho_{i+1}})}|G|,$$

by the first orthogonality relation for characters (χ = character of V^{ρ_i}). Since the trace of τ_{i,V_α} is

$$\frac{\dim(V^{\rho_1})\cdots\dim(V^{\rho_n})}{\dim(V^{\rho_i})},$$

it follows that $\beta = \frac{|G|}{\dim(V^{\rho_i})}$. We have

$$\rho_i = \rho_{i+1} \quad \text{implies} \quad b_i v = \frac{|G|}{\dim(V^{\rho_i})}\tau_{i,V_\alpha}(v),\ v \in V_\alpha. \tag{27}$$

The following result relates the action of s_i on GZ-subspaces to transformations on the corresponding weights.

Theorem 5.3 *Let $\alpha = ((\rho_1, \ldots, \rho_n), \alpha_1, \ldots, \alpha_n) \in spec_G(n, \lambda)$ and consider the GZ-subspace V_α of V^λ. Then*

(i) For $1 \leq i \leq n-1$, $s_i \cdot V_\alpha = V_\alpha$ iff $\rho_i = \rho_{i+1}$ and $\alpha_{i+1} = \alpha_i \pm \frac{|G|}{dim(V^{\rho_i})}$.

(ii) For $1 \leq i \leq n-1$ we have

(a) $\rho_i = \rho_{i+1}$ and $\alpha_{i+1} = \alpha_i + \frac{|G|}{dim(V^{\rho_i})}$ implies $s_i v = \tau_{i,V_\alpha}(v)$, $v \in V_\alpha$.

(b) $\rho_i = \rho_{i+1}$ and $\alpha_{i+1} = \alpha_i - \frac{|G|}{dim(V^{\rho_i})}$ implies $s_i v = -\tau_{i,V_\alpha}(v)$, $v \in V_\alpha$.

(iii) If $\rho_i = \rho_{i+1}$ then $\alpha_i \neq \alpha_{i+1}$, $1 \leq i \leq n-1$.

(iv) For $i = 1, \ldots, n-2$ the following statements are not true.

(a) $\rho_i = \rho_{i+1} = \rho_{i+2}$ and $\alpha_i = \alpha_{i+1} + \frac{|G|}{dim(V^{\rho_i})} = \alpha_{i+2}$.

(b) $\rho_i = \rho_{i+1} = \rho_{i+2}$ and $\alpha_i = \alpha_{i+1} - \frac{|G|}{dim(V^{\rho_i})} = \alpha_{i+2}$.

(v) For $1 \leq i \leq n-1$, if $\rho_i \neq \rho_{i+1}$ then $U = s_i \cdot V_\alpha$ is a GZ-subspace of V^λ with weight

$$s_i \cdot \alpha = ((\rho_1, \ldots, \rho_{i-1}, \rho_{i+1}, \rho_i, \rho_{i+2}, \ldots, \rho_n), \alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \alpha_i, \alpha_{i+2}, \ldots, \alpha_n).$$

(vi) For $1 \leq i \leq n-1$, if $\rho_i = \rho_{i+1}$ and $\alpha_{i+1} \neq \alpha_i \pm \frac{|G|}{dim(V^{\rho_i})}$ then, setting

$$U = \left(s_i - \frac{|G|}{(\alpha_{i+1} - \alpha_i) dim(V^{\rho_i})} \tau_{i,V_\alpha} \right) V_\alpha,$$

we have that U is a GZ-subspace V^λ with weight

$$s_i \cdot \alpha = ((\rho_1, \ldots, \rho_n), \alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \alpha_i, \alpha_{i+2}, \ldots, \alpha_n).$$

Proof (i) (only if) That $\rho_i = \rho_{i+1}$ is clear from (22). The maps $V_\alpha \to V_\alpha$ given by $v \mapsto s_i v$ and $v \mapsto \tau_{i,V_\alpha}(v)$ are both involutions. Therefore the possible eigenvalues are 1 and -1.

Let $u, v \in V_\alpha, u, v \neq 0$ with $s_i v = v$ and $s_i u = -u$. Then

$$\alpha_i v = X_i(v) = X_i(s_i v) = (s_i X_{i+1} - b_i) v = \alpha_{i+1} v - \frac{|G|}{\dim(V^{\rho_i})} \tau_{i,V_\alpha}(v), \tag{28}$$

$$\alpha_i u = X_i(u) = X_i(-s_i u) = -(s_i X_{i+1} - b_i) u = \alpha_{i+1} u + \frac{|G|}{\dim(V^{\rho_i})} \tau_{i,V_\alpha}(u). \tag{29}$$

It follows that $\tau_{i,V_\alpha}(v)$ is a multiple of v and is thus either v or $-v$. Similarly, $\tau_{i,V_\alpha}(u)$ is a multiple of u and is thus either u or $-u$. Since there exists an eigenvector of $s_i : V_\alpha \to V_\alpha$, the result follows.

(if) Since $s_i V_\alpha$ is also an irreducible G^n-module (by (22)) either $V_\alpha = s_i V_\alpha$ or $s_i V_\alpha \cap V_\alpha = \{0\}$. Assume that $s_i V_\alpha \cap V_\alpha = \{0\}$. We shall derive a contradiction.

We assume that $\alpha_{i+1} = \alpha_i + \frac{|G|}{\dim(V^{\rho_i})}$. The case $\alpha_{i+1} = \alpha_i - \frac{|G|}{\dim(V^{\rho_i})}$ is similar.

The subspace $V_\alpha \oplus s_i V_\alpha$ of V^λ is an A_i-module, by Lemma 5.1. Define a subspace

$$M \quad = \quad \{v - s_i \cdot \tau_{i,V_\alpha}(v) | v \in V_\alpha\} \subseteq V_\alpha \oplus s_i V_\alpha.$$

We check that M is an A_i-submodule, i.e., is closed under the action of s_i, X_i, X_{i+1}, and $g^{(l)}, l = 1, \ldots, n$. We have, for $v \in V_\alpha$,

$$\begin{aligned}
s_i(v - s_i\tau_{i,V_\alpha}(v)) &= s_i v - \tau_{i,V_\alpha}(v) \\
&= -\tau_{i,V_\alpha}(v) - s_i(\tau_{i,V_\alpha}(-\tau_{i,V_\alpha}(v))) \in M, \\
g^{(l)}(v - s_i\tau_{i,V_\alpha}(v)) &= g^{(l)}v - g^{(l)}s_i\tau_{i,V_\alpha}(v) \\
&= g^{(l)}v - s_i g^{(\omega_i(l))}\tau_{i,V_\alpha}(v) \\
&= g^{(l)}v - s_i\tau_{i,V_\alpha}(g^{(l)}v) \in M, \\
X_i(v - s_i\tau_{i,V_\alpha}(v)) &= \alpha_i v - (s_i X_{i+1} - b_i)(\tau_{i,V_\alpha}(v)) \\
&= \alpha_i v - \alpha_{i+1} s_i \tau_{i,V_\alpha}(v) + \frac{|G|}{\dim(V^{\rho_i})} v \\
&= \alpha_{i+1}(v - s_i\tau_{i,V_\alpha}(v)) \in M, \\
X_{i+1}(v - s_i\tau_{i,V_\alpha}(v)) &= \alpha_{i+1} v - (s_i X_i + b_i)(\tau_{i,V_\alpha}(v)) \\
&= \alpha_{i+1} v - \alpha_i s_i \tau_{i,V_\alpha}(v) - \frac{|G|}{\dim(V^{\rho_i})} v \\
&= \alpha_i(v - s_i\tau_{i,V_\alpha}(v)) \in M.
\end{aligned}$$

We shall now show that M is the only nonempty, proper A_i-submodule of $V_\alpha \oplus s_i V_\alpha$. Since $\dim(M) < \dim(V_\alpha \oplus s_i V_\alpha)$, this contradicts the fact that $V_\alpha \oplus s_i V_\alpha$ is a semisimple A_i-module (since A_i is a semisimple algebra by Lemma 5.2).

Let M' be a nonempty, proper A_i-submodule of $V_\alpha \oplus s_i V_\alpha$. Since M' is closed under s_i we have $M' \not\subseteq V_\alpha$ and $M' \not\subseteq s_i V_\alpha$. Also, M' is in particular a G^n-submodule of $V_\alpha \oplus s_i V_\alpha$. Since V_α and $s_i V_\alpha$ are isomorphic irreducible G^n-modules and $v \mapsto s_i\tau_{i,V_\alpha}(v)$ is a G^n-linear isomorphism between them, it follows by Schur's lemma that

$$M' \quad = \quad \{v + \gamma s_i \tau_{i,V_\alpha}(v) | v \in V_\alpha\},$$

for some $0 \neq \gamma \in \mathbb{C}$. We shall show that $\gamma = -1$.

Now

$$s_i(v + \gamma s_i\tau_{i,V_\alpha}(v)) = s_i v + \gamma\tau_{i,V_\alpha}(v) = \gamma\tau_{i,V_\alpha}(v) + \frac{\gamma s_i \tau_{i,V_\alpha}(\gamma\tau_{i,V_\alpha}(v))}{\gamma^2},$$

which yields $\gamma^2 = 1$ and hence $\gamma = \pm 1$.

We have

$$\begin{aligned}
X_i(v + s_i\tau_{i,V_\alpha}(v)) &= \alpha_i v + \alpha_{i+1} s_i \tau_{i,V_\alpha}(v) - b_i \tau_{i,V_\alpha}(v) \\
&= \left(\alpha_i - \frac{|G|}{\dim(V^{\rho_i})}\right) v + \alpha_{i+1} s_i \tau_{i,V_\alpha}(v), \\
X_i(v - s_i\tau_{i,V_\alpha}(v)) &= \alpha_i v - \alpha_{i+1} s_i \tau_{i,V_\alpha}(v) + b_i \tau_{i,V_\alpha}(v) \\
&= \left(\alpha_i + \frac{|G|}{\dim(V^{\rho_i})}\right) v - \alpha_{i+1} s_i \tau_{i,V_\alpha}(v).
\end{aligned}$$

and thus $\gamma = -1$. Thus $M' = M$ and the proof of the if part is complete.

(ii) This follows from (28) and (29).

(iii) Either $s_iV_\alpha = V_\alpha$ or $s_iV_\alpha \cap V_\alpha = \{0\}$. If $s_iV_\alpha = V_\alpha$ then by part (i) $\alpha_{i+1} = \alpha_i \pm \frac{|G|}{\dim(V^{\rho_i})}$, so $\alpha_i \neq \alpha_{i+1}$.

Now assume $s_iV_\alpha \cap V_\alpha = \{0\}$. Then, as checked before, $V_\alpha \oplus s_iV_\alpha$ is A_i-invariant. Choose a basis B of V_α and consider the basis $B \cup s_iB$ of $V_\alpha \oplus s_iV_\alpha$. Let N be the matrix of τ_{i,V_α} with respect to the basis B and set $\kappa = \frac{|G|}{\dim(V^{\rho_i})}$. Using the relation $X_is_i = s_iX_{i+1} - b_i$ we see that the matrices of X_i and X_{i+1} (respectively) with respect to $B \cup sB$ are given in block form as follows

$$\begin{bmatrix} \alpha_i I & -\kappa N \\ 0 & \alpha_{i+1} I \end{bmatrix}, \quad \begin{bmatrix} \alpha_{i+1} I & \kappa N \\ 0 & \alpha_i I \end{bmatrix}$$

The actions of X_i and X_{i+1} on $V_\alpha \oplus s_iV_\alpha$ are diagonalizable by Lemma 5.2 and thus $\alpha_i \neq \alpha_{i+1}$ (since $N \neq 0$).

(iv) Suppose statement (a) is true. Since, as a G^n-module, V_α is isomorphic to $V^{\rho_1} \otimes \cdots \otimes V^{\rho_n}$, we can choose a $v \in V_\alpha$ such that $\tau_{i,V_\alpha}(v) = \tau_{i+1,V_\alpha}(v) = v$. By part (ii) we have $s_iv = -v$ and $s_{i+1}v = v$. Now consider the Coxeter relation

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$$

and let both sides act on v. The left hand side yields v and the right hand side yields $-v$, a contradiction. So, statement (a) must be false. The proof for the falsity of statement (b) is similar.

(v) When $\rho_i \neq \rho_{i+1}$ then, by (26), $b_iv = 0$ for $v \in V_\alpha$. The result now follows from (22) and the relation $X_is_i = s_iX_{i+1} - b_i$.

(vi) By part (i) $s_iV_\alpha \cap V_\alpha = \{0\}$ and by part (iii) $\alpha_i \neq \alpha_{i+1}$. Clearly, U is a subspace of $V_\alpha \oplus s_iV_\alpha$ and since $v \mapsto s_i\tau_{i,V_\alpha}(v)$ (or, equivalently, $\tau_{i,V_\alpha}(v) \mapsto s_iv$) is a G^n-linear isomorphism between V_α and s_iV_α it follows that U is also an irreducible G^n-module with label $(\rho_1, \ldots, \rho_n)$. It remains to check that X_i, X_{i+1} act on U by appropriate scalars.

Setting $\kappa = \frac{|G|}{\dim(V^{\rho_i})}$, we have, using (27),

$$\begin{aligned} X_i\left(s_i v - \frac{\kappa}{\alpha_{i+1}-\alpha_i}\tau_{i,V_\alpha}(v)\right) &= \alpha_{i+1}(s_i v) - b_i v - \frac{\alpha_i \kappa}{\alpha_{i+1}-\alpha_i}\tau_{i,V_\alpha}(v) \\ &= \alpha_{i+1}\left(s_i v - \frac{\kappa}{\alpha_{i+1}-\alpha_i}\tau_{i,V_\alpha}(v)\right), \\ X_{i+1}\left(s_i v - \frac{\kappa}{\alpha_{i+1}-\alpha_i}\tau_{i,V_\alpha}(v)\right) &= \alpha_i(s_i v) + b_i v - \frac{\alpha_{i+1} \kappa}{\alpha_{i+1}-\alpha_i}\tau_{i,V_\alpha}(v) \\ &= \alpha_i\left(s_i v - \frac{\kappa}{\alpha_{i+1}-\alpha_i}\tau_{i,V_\alpha}(v)\right). \end{aligned}$$

That completes the proof. □

Let $\alpha = ((\rho_1, \ldots, \rho_n), \alpha_1, \ldots, \alpha_n) \in \mathrm{spec}_G(n)$. We say that the transposition s_i is *admissible* for α if one of the following conditions holds:

(i) $\rho_i \neq \rho_{i+1}$ or

(ii) $\rho_i = \rho_{i+1}$ and $\alpha_i \neq \alpha_{i+1} \pm \frac{|G|}{\dim(V^{\rho_i})}$.

The following two observations are easy to see:

(a) For $\alpha, \beta \in \mathrm{spec}_G(n)$, we have $\alpha \sim \beta$ if α is obtained from β by a sequence of admissible transpositions.

(b) We have

$$\begin{aligned} &((\rho_1, \ldots, \rho_n), \alpha_1, \ldots, \alpha_n) \in \mathrm{spec}_G(n) \text{ implies} \\ &\qquad ((\rho_1, \ldots, \rho_{n-1}), \alpha_1, \ldots, \alpha_{n-1}) \in \mathrm{spec}_G(n-1). \end{aligned} \tag{30}$$

6 Content vectors and Young G-tableaux

In the Vershik-Okounkov theory Young G-tableaux are related to irreducible representations of G_n via their content vectors. Let us define these first.

Let $\alpha = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n$. We say that α is a *content vector* if

(i) $a_1 = 0$.

(ii) $\{a_i - 1, a_i + 1\} \cap \{a_1, a_2, \ldots, a_{i-1}\} \neq \emptyset$, for all $i > 1$.

(iii) if $a_i = a_j = a$ for some $i < j$ then $\{a-1, a+1\} \subseteq \{a_{i+1}, \ldots, a_{j-1}\}$ (i.e., between two occurrences of a there should also be occurrences of $a-1$ and $a+1$).

Condition (ii) in the definition above can be replaced (in the presence of conditions (i) and (iii)) by condition (ii') below.

(ii') For all $i > 1$, if $a_i > 0$ then $a_j = a_i - 1$ for some $j < i$ and if $a_i < 0$ then $a_j = a_i + 1$ for some $j < i$.

The set of all content vectors of length n is denoted $\mathrm{cont}(n) \subseteq \mathbb{Z}^n$. It is convenient to assume that the empty sequence is a content vector of length 0 and is the unique element of $\mathrm{cont}(0)$.

Let $\alpha = ((\rho_1, \ldots, \rho_n), a_1, \ldots, a_n)$, where $\rho_i \in G^\wedge$ for all i, and $(a_1, \ldots, a_n) \in \mathbb{C}^n$. For $\sigma \in G^\wedge$, let $\sigma(J) = \{j_1 < j_2 < \cdots < j_{n_{\sigma,\alpha}}\} \subseteq \{1, 2, \ldots, n\}$ be the set of indices satisfying $\rho_{j_i} = \sigma$, $i = 1, \ldots, n_{\sigma,\alpha}$ and $\rho_l \neq \sigma$ for $l \in \{1, 2, \ldots, n\} - \sigma(J)$. Let $\sigma(\alpha)$ be the sequence

$$\left(\frac{\dim(V^\sigma)}{|G|} a_{j_1}, \ldots, \frac{\dim(V^\sigma)}{|G|} a_{j_{n_{\sigma,\alpha}}}\right).$$

We say that α is a *content vector with respect to G of length n* if $\sigma(\alpha) \in \mathrm{cont}(n_{\sigma,\alpha})$ for all $\sigma \in G^\wedge$. Since $\dim(V^\sigma)$ divides $|G|$ it follows that $a_i \in \mathbb{Z}$ for all i. Denote by $\mathrm{cont}_G(n) \subseteq \mathbb{Z}^n$ the set of all content vectors with respect to G of length n.

Theorem 6.1 *We have $spec_G(n) \subseteq cont_G(n)$.*

Proof Let $\alpha = (\rho, a_1, \ldots, a_n) \in \mathrm{spec}_G(n)$, where $\rho = (\sigma, \ldots, \sigma)$, $\sigma \in G^\wedge$. We will show that

$$\left(\frac{\dim(V^\sigma)}{|G|} a_1, \ldots, \frac{\dim(V^\sigma)}{|G|} a_n\right) \in \mathrm{cont}(n).$$

Using Theorem 5.3(v) and (30) we see that this proves the result.

Clearly $a_1 = 0$ as $X_1 = 0$. We verify conditions (ii) and (iii) in the definition of content vectors by induction on n. Since $X_2 = b_1$ we have, from (27), that $\frac{\dim(V^\sigma)}{|G|} a_2 = \pm 1$ and thus condition (ii) is verified (and condition (iii) does not apply). Now assume $n \geq 3$.

We first verify condition (ii). If $a_{n-1} = a_n \pm \frac{|G|}{\dim(V^\sigma)}$ there is nothing to prove, so assume this does not hold. Then the transposition $(n-1, n)$ is admissible for α and, by Theorem 5.3(vi), $(\rho, a_1, \ldots, a_{n-2}, a_n, a_{n-1}) \in \mathrm{spec}_G(n)$. Now, by (30) and the induction hypothesis, $\{a_n - \frac{|G|}{\dim(V^\sigma)}, a_n + \frac{|G|}{\dim(V^\sigma)}\} \cap \{a_1, \ldots, a_{n-2}\} \neq \emptyset$. Thus condition (ii) is verified.

We now verify condition (iii). Now assume that $a_i = a_n = a$ for some $i < n$. We may assume that i is the largest possible index, i.e., a does not occur between a_i and a_n, so $a \notin \{a_{i+1}, \ldots, a_{n-1}\}$. Now assume that $a - \frac{|G|}{\dim(V^\sigma)} \notin \{a_{i+1}, \ldots, a_{n-1}\}$. We shall derive a contradiction (the case where $a + \frac{|G|}{\dim(V^\sigma)} \notin \{a_{i+1}, \ldots, a_{n-1}\}$ is similar).

By induction hypothesis the number $a + \frac{|G|}{\dim(V^\sigma)}$ occurs in $\{a_{i+1}, \ldots, a_{n-1}\}$ at most once (for, if it occured twice, then by the induction hypothesis a would also occur contradicting our choice of i). Thus there are two possibilities:

$$(a_i, \ldots, a_n) = (a, *, \ldots, *, a) \text{ or } (a_i, \ldots, a_n) = (a, *, \ldots, *, a + \frac{|G|}{\dim(V^\sigma)}, *, \ldots, *, a),$$

where $*$ stands for a number different from $a - \frac{|G|}{\dim(V^\sigma)}, a, a + \frac{|G|}{\dim(V^\sigma)}$.

In the first case we can apply a sequence of admissible transpositions to infer that $(\rho, \ldots, a, a, \ldots) \in \mathrm{spec}_G(n)$, contradicting Theorem 5.3(iii) and in the second case we can apply a sequence of admissible transpositions to infer that $(\rho, \ldots, a, a + \frac{|G|}{\dim(V^\sigma)}, a, \ldots) \in \mathrm{spec}_G(n)$, contradicting Theorem 5.3(iv)(b). $\square$

Let $\alpha = ((\rho_1, \ldots, \rho_n), a_1, \ldots, a_n) \in \mathrm{cont}_G(n)$ We say that the transposition s_i is *admissible* for α if $\rho_i \neq \rho_{i+1}$, or $\rho_i = \rho_{i+1}$ and $a_i \neq a_{i+1} \pm \frac{|G|}{\dim(V^{\rho_i})}$. We define the following equivalence relation on $\mathrm{cont}_G(n)$: $\alpha \approx \beta$ if β can be obtained from α by a sequence of (zero or more) admissible transpositions.

We now introduce Young G-tableaux into the picture.

Let $T_1 \in \mathrm{tab}_G(n)$ and assume that either i and $i+1$ do not appear in the same Young diagram of T_1 or that they are in the same Young diagram of T_1 but do not appear in the same row or same column of this Young diagram. Then exchanging i and $i+1$ in T_1 produces another standard Young G-tableau $T_2 \in \mathrm{tab}_G(n)$. We say that T_2 is obtained from T_1 by an *admissible transposition*. For $T_1, T_2 \in \mathrm{tab}_G(n)$, define $T_1 \approx T_2$ if T_2 can be obtained from T_1 by a sequence of (zero or more) admissible transpositions (it is easily seen that $\approx$ is an equivalence relation).

Lemma 6.2 *Let $T_1, T_2 \in tab_G(n)$. Then $T_1 \approx T_2$ if and only T_1 and T_2 have the same shape.*

Proof The only if part is obvious. To prove the if part we proceed as follows. Let $\mu \in \mathcal{Y}_n(G^\wedge)$. Enumerate the elements of $G^\wedge$ as $\sigma_1, \ldots, \sigma_t$. Let n_i be the number of boxes in the Young diagram $\mu(\sigma_i)$. Then $n_1 + \cdots + n_t = n$.

Define the following element R of $\mathrm{tab}_G(n, \mu)$: fill the Young diagram of $\mu(\sigma_1)$ with the numbers $1, 2, \ldots, n_1$ in *row major order*, i.e., the first row with the numbers $1, 2, \ldots, l_1$ (in increasing order, here l_1 = length of first row), the second row with $l_1 + 1, \ldots, l_1 + l_2$ (in increasing order, here l_2 = length of second row) and so on till the last row of $\mu(\sigma_1)$. Now fill the Young diagram of $\mu(\sigma_2)$ with the numbers $n_1 + 1, \ldots, n_1 + n_2$ in row major order and so on till the last Young diagram $\mu(\sigma_t)$.

We show that any $T \in \mathrm{tab}_G(n, \mu)$ satisfies $T \approx R$. This will prove the if part. Consider the last box of the last row of the last Young diagram $\mu(\sigma_t)$. Let i be written in this box of T. Exchange i and $i+1$ in T (which is clearly an admissible transposition). Now repeat this procedure with $i+1$ and $i+2$, then $i+2$ and $i+3$, and finally $n-1$ and n. At the end of this sequence of admissible transpositions we have the number n written in the last box of the last row of $\mu(\sigma_t)$. Now repeat the same procedure for $n-1, n-2, \ldots, 2$. $\square$

Let us make a remark about the proof of Lemma 6.2. Let s denote the permutation that maps R to T. Then the proof shows that R can be obtained from T by a sequence of $\ell(s)$ admissible transpositions. Thus T can be obtained from R by a sequence of $\ell(s)$ admissible transpositions.

The *content* $c(b)$ of a box b of a Young diagram is its y-coordinate $-$ its x-coordinate

(our convention for drawing Young diagrams is akin to writing down matrices with x-axis running downwards and y axis running to the right).

Lemma 6.3 *Let $\Phi : tab_G(n) \to cont_G(n)$ be defined as follows. Given $T \in tab_G(n)$ and $1 \le i \le n$, let $b_T(i)$ be the box (in one of the Young diagrams of T) where the number i resides. Define*

$$\Phi(T) = ((r_T(1), \ldots, r_T(n)),\ \frac{|G|}{\dim(V^{r_T(1)})}\, c(b_T(1)), \ldots,\ \frac{|G|}{\dim(V^{r_T(n)})}\, c(b_T(n))).$$

Then Φ is a bijection which takes $\approx$-equivalent standard Young G-tableaux to $\approx$-equivalent content vectors with respect to G.

Proof The general case clearly follows from the $|G| = 1$ case for which we need to give a bijection between content vectors of length n and standard Young tableaux with n boxes. This is well known. The content vector of any standard Young tableau clearly satisfies conditions (i), (ii), and (iii) in the definition of a content vector and these conditions uniquely determine the numbers to be filled in the boxes of the Young diagram. This bijection clearly preserves the $\approx$ relation. $\Box$

Theorem 6.4 *(i) $spec_G(n) = cont_G(n)$ and the equivalence relations $\sim$ and $\approx$ coincide.*

(ii) The map $\Phi^{-1} : spec_G(n) \to tab_G(n)$ is a bijection and, for $\alpha, \beta \in spec_G(n)$, we have $\alpha \sim \beta$ if and only if $\Phi^{-1}(\alpha)$ and $\Phi^{-1}(\beta)$ have the same shape.

Proof We have

(a) $\mathrm{spec}_G(n) \subseteq \mathrm{cont}_G(n)$.

(b) If $\alpha \in \mathrm{spec}_G(n)$, $\beta \in \mathrm{cont}_G(n)$, and $\alpha \approx \beta$ then it is easily seen that $\beta \in \mathrm{spec}_G(n)$ and $\alpha \sim \beta$. It follows that given an $\sim$-equivalence class $\mathcal{A}$ of $\mathrm{spec}_G(n)$ and an $\approx$-equivalence class $\mathcal{B}$ of $\mathrm{cont}_G(n)$, either $\mathcal{A} \cap \mathcal{B} = \emptyset$ or $\mathcal{B} \subseteq \mathcal{A}$.

(c) $|(\mathrm{spec}_G(n)/\sim)| = |G_n^\wedge| = |\mathcal{P}_n(G_*)| = |\mathcal{Y}_n(G^\wedge)|$, since the number of irreducible G_n-representations is equal to the number of conjugacy classes in G_n and similarly for G.

(d) $|(\mathrm{cont}_G(n)/\approx)| = |\mathcal{Y}_n(G^\wedge)|$, by Lemmas 6.3 and 6.2.

The four statements above imply part (i). Part (ii) is now clear. $\Box$

Using Theorem 6.4 we may parametrize the irreducible representations of G_n by elements of $\mathcal{Y}_n(G^\wedge)$. The following result is a reformulation of the GZ-decomposition in terms of standard Young G-tableaux.

Theorem 6.5 *Let $\mu \in \mathcal{Y}_n(G^\wedge)$. Then we may index the GZ-subspaces of V^μ by standard Young G-tableaux of shape μ and write the GZ-decomposition (17) as*

$$V^\mu = \oplus_{T \in tab_G(n,\mu)} V_T, \tag{31}$$

where each V_T is closed under the action of $G^n = G \times \cdots \times G$ (n factors) and, as a G^n-module, is isomorphic to the irreducible G^n-module

$$V^{r_T(1)} \otimes V^{r_T(2)} \otimes \cdots \otimes V^{r_T(n)}.$$

For $i = 1, \ldots, n$, the eigenvalue of X_i on V_T is given by $\frac{|G|}{\dim(V^{r_T(i)})}\, c(b_T(i))$. □

The branching rule for the pair $G_{n-1} \subseteq G_n$ is now clear.

Theorem 6.6 *Let $\mu \in \mathcal{Y}_{n+1}(G^\wedge)$. Then we have a G_n-module isomorphism*

$$V^\mu \;\cong\; \oplus_{\sigma \in G^\wedge} \dim(V^\sigma) \left(\oplus_{\lambda \in \mu \downarrow \sigma} V^\lambda\right). \;\square$$

The dimension of an irreducible representation of G_n easily follows from Theorem 6.5. For a Young diagram μ let f^μ denote the number of standard Young tableaux of shape μ.

Theorem 6.7 *Let $\mu \in \mathcal{Y}_n(G^\wedge)$. Write the elements of $G^\wedge$ as $\{\sigma_1, \ldots, \sigma_t\}$ and set*

$$\mu_i = \mu(\sigma_i),\; n_i = |\mu_i|,\; d_i = \dim(V^{\sigma_i}),\; i = 1, \ldots, t.$$

Then

$$\dim(V^\mu) = \binom{n}{n_1, \ldots, n_t} f^{\mu_1} \cdots f^{\mu_t}\, d_1^{n_1} \cdots d_t^{n_t}. \square$$

We now discuss the choice of a basis of V^μ, $\mu \in \mathcal{Y}_n(G^\wedge)$, with respect to which we may write down the matrices for the action of the Coxeter generators $s_1, \ldots, s_{n-1}$. We begin with an observation.

Fix $\mu \in \mathcal{Y}_n(G^\wedge)$ and consider the irreducible G_n-module V^μ. Let $T \in \mathrm{tab}_G(n, \mu)$ and let p_T denote the projection of V^μ onto V_T determined by the decomposition (31). Let s_i be an admissible transposition for T. Two cases arise:

(a) i and $i+1$ are in different Young diagrams of T: It follows from Theorem 5.3 (v) that $p_{s_i \cdot T}(s_i \cdot B)$ is a basis of $V_{s_i \cdot T}$ for any basis B of V_T.

(b) i and $i+1$ are in the same Young diagram of T but are not in the same row or the same column of this Young diagram: Let $0 \neq v \in V_T$. It follows from (the proof of) Theorem 5.3 (vi) that $s_i \cdot v$ is the sum of a nonzero rational multiple of $\tau_{i,V_T}(v)$ and a nonzero vector in $V_{s_i \cdot T}$ and that the map $V_T \to V_{s_i \cdot T}$ given by $v \mapsto p_{s_i \cdot T}(s_i \cdot v)$ is a linear isomorphism. In particular, $p_{s_i \cdot T}(s_i \cdot B)$ is a basis of $V_{s_i \cdot T}$ for any basis B of V_T.

Now let R be the tableau defined in the proof of Lemma 6.2. Fix a basis B_R of V_R. Consider a standard G-tableau $T \in \mathrm{tab}_G(n, \mu)$. Let s be the permutation that maps R to T. Define

$$B_T \;=\; \{p_T(s \cdot v) \mid v \in B_R\}, \tag{32}$$

and define $\ell(T)$, the *length* of T, to be $\ell(s)$. The following result now easily follows, by induction on the length of $T \in \mathrm{tab}_G(n, \mu)$, using observations (a), (b) above and the fact, remarked after the proof of Lemma 6.2, that $T \in \mathrm{tab}_G(n, \mu)$ can be obtained from R by a sequence of $\ell(T)$ admissible transpositions and no fewer Coxeter transpositions.

Lemma 6.8 *(i) B_T is a basis of V_T, for all $T \in tab_G(n,\mu)$.*

(ii) Let $T \in tab_G(n,\mu)$ and let s_i be an admissible transposition for T. Then

(a) If i and $i+1$ are in different Young diagrams of T we have

$$B_{s_i \cdot T} = \{p_{s_i \cdot T}(s_i \cdot v) \mid v \in B_T\}.$$

(b) If i and $i+1$ are in the same Young diagram of T but not in the same row or same column of this Young diagram we have

$$\begin{array}{rcl} B_{s_i \cdot T} & = & \{p_{s_i \cdot T}(s_i \cdot v) \mid v \in B_T\}, \;\; \text{if } \ell(s_i \cdot T) = \ell(T)+1, \\ B_{s_i \cdot T} & = & \{(1-r^{-2})p_{s_i \cdot T}(s_i \cdot v) \mid v \in B_T\}, \;\; \text{if } \ell(s_i \cdot T) = \ell(T)-1, \end{array}$$

where $r = \frac{(c(b_T(i+1)) - c(b_T(i)))}{|G|}\, dim(V^{r_T(i)})$.

We now choose a basis of V_R in a certain way and then apply the method above to get bases of all the GZ-subspaces. For $\sigma \in G^\wedge$, fix a basis B^σ of V^σ. Then, for $\rho = (\rho_1, \ldots, \rho_n)$, where $\rho_i \in G^\wedge$ for all i, we have that $B^\rho = B^{\rho_1} \otimes \cdots \otimes B^{\rho_n}$ is a basis of $V^\rho = V^{\rho_1} \otimes \cdots \otimes V^{\rho_n}$. Thus, for $T \in \mathrm{tab}_G(n,\mu)$, we have that B^{r_T} is a basis of V^{r_T}, where $r_T = (r_T(1), \ldots, r_T(n))$. Let N_{i,r_T} be the matrix of the switch operator τ_{i,r_T} on V^{r_T} with respect to the basis B^{r_T}.

Let R be the standard G-tableau defined above and fix a G-linear isomorphism

$$f : V^{r_R(1)} \otimes \cdots \otimes V^{r_R(n)} \to V_R.$$

Define the basis

$$\overline{B_R} = f(B^{r_R(1)} \otimes \cdots \otimes B^{r_R(n)})$$

of V_R. Now use (32) to define a basis $\overline{B_T}$ of V_T for all $T \in \mathrm{tab}_G(n,\mu)$.

Let $T \in \mathrm{tab}_G(n,\mu)$ and s be the permutation that maps R to T. Now S_n acts on V^{r_T} by permuting the coordinates and the image of the action of s^{-1} on V^{r_T} is V^{r_R}. The following result now follows.

Lemma 6.9 *The map $V^{r_T} \to V_T$ given by $v \mapsto p_T(sfs^{-1}(v))$ is a G^n-linear isomorphism that takes the basis B^{r_T} of V^{r_T} to the basis $\overline{B_T}$ of V_T. Thus the matrix $N_{i,T}$ of τ_{i,V_T} with respect to $\overline{B_T}$ is equal to N_{i,r_T}.*

We now have the following result.

Theorem 6.10 *Consider the basis $\cup_{T \in tab_G(n,\mu)} \overline{B_T}$ of V^μ. Fix $T \in tab_G(n,\mu)$ and let $\Phi(T) = ((\rho_1, \ldots, \rho_n), a_1, \ldots, a_n)$. Let s_i be a Coxeter generator. Let I denote the $|\overline{B_T}| \times |\overline{B_T}|$ identity matrix. Set $r = \frac{(a_{i+1} - a_i) dim(V^{\rho_i})}{|G|}$ and $N = N_{i,T}$.*

The action of s_i on V_T is as follows.

(i) If i and $i+1$ are in the same column of the same Young diagram of T then V_T is closed under the action of s_i and the matrix of this action with respect to the basis $\overline{B_T}$ is N.

(ii) If i and $i+1$ are in the same row of the same Young diagram of T then V_T is closed under the action of s_i and the matrix of this action with respect to the basis $\overline{B_T}$ is $-N$.

(iii) Suppose i and $i+1$ are not in the same Young diagram of T. Let $S = s_i \cdot T$. Then $V_T \oplus V_S$ is closed under the action of s_i and the matrix of this action, with respect to the basis $\overline{B_T} \cup \overline{B_S}$, is given by

$$\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}.$$

(iv) Suppose i and $i+1$ are in the same Young diagram of T but not in the same row or same column of this Young diagram. Let $S = s_i \cdot T$. Then $N = N_{i,S}$.

If $\ell(S) = \ell(T) + 1$ then $V_T \oplus V_S$ is closed under the action of s_i and the matrix of this action, with respect to the basis $\overline{B_T} \cup \overline{B_S}$, is given by

$$\begin{bmatrix} r^{-1}N & (1-r^{-2})I \\ I & -r^{-1}N \end{bmatrix}.$$

If $\ell(S) = \ell(T) - 1$ then the matrix of the action of s_i on the subspace $V_T \oplus V_S$ with respect to the basis basis $\overline{B_T} \cup \overline{B_S}$ is given by the transpose of the matrix above.

Proof Parts (i), (ii), (iii) and part (iv) with $\ell(S) = \ell(T) + 1$ follow from Theorem 5.3, Lemma 6.8, and Lemma 6.9 above. To prove the case $\ell(S) = \ell(T) - 1$ of part (iv), switch T and S in the $\ell(S) = \ell(T) + 1$ case along with switching a_i and a_{i+1}. This is equivalent to transposing the matrix. □

The basis of V^μ and the action of s_i described above correspond to Young's *seminormal form* in the case of the symmetric groups. Now let us consider the analog of Young's *orthogonal form*. Since V^μ is irreducible there is a unique (upto scalars) G_n-invariant inner product on V^μ. Choose and fix one such inner product. Since the branching from $H_{i,n}$ to $H_{i-1,n}$ is multiplicity free we have that the decomposition of an irreducible $H_{i,n}$-module into irreducibles $H_{i-1,n}$-modules is orthogonal. It follows that the GZ-decomposition (31) of V^μ is orthogonal.

For $\sigma \in G^\wedge$, fix a G-invariant inner product (unique upto scalars) on V^σ, and fix an orthonormal basis C^σ of V^σ. Then, for $\rho = (\rho_1, \ldots, \rho_n)$, where $\rho_i \in G^\wedge$ for all i, we have that $C^\rho = C^{\rho_1} \otimes \cdots \otimes C^{\rho_n}$ is an orthonormal basis of $V^\rho = V^{\rho_1} \otimes \cdots \otimes V^{\rho_n}$ (under the inner product obtained by multiplying the component inner products). Thus, for $T \in \mathrm{tab}_G(n, \mu)$, we have that C^{r_T} is an orthonormal basis of V^{r_T}, where $r_T = (r_T(1), \ldots, r_T(n))$. Let M_{i,r_T} be the matrix of the switch operator τ_{i,r_T} on V^{r_T} with respect to the basis C^{r_T}.

Let R be the standard G-tableau defined above and fix a G^n-linear isometry

$$f : V^{r_R(1)} \otimes \cdots \otimes V^{r_R(n)} \to V_R.$$

Define the orthonormal basis

$$C_R = f(C^{r_R(1)} \otimes \cdots \otimes C^{r_R(n)})$$

of V_R. Now use (32) to define a basis C_T of V_T for all $T \in \mathrm{tab}_G(n, \mu)$.

Let $T \in \mathrm{tab}_G(n, \mu)$ and s be the permutation that maps R to T. We have

Lemma 6.11 *The map $V^{r_T} \to V_T$ given by $v \mapsto p_T(sfs^{-1}(v))$ is a G^n-linear isometry that takes the basis C^{r_T} of V^{r_T} to the basis C_T of V_T. Thus the matrix $M_{i,T}$ of τ_{i,V_T} with respect to C_T is equal to M_{i,r_T}.*

The following result can be proved along the lines of the previous result.

Theorem 6.12 *Consider the orthonormal basis $\cup_{T \in tab_G(n,\mu)} C_T$ of V^μ defined above. Fix $T \in tab_G(n, \mu)$ and let $\Phi(T) = ((\rho_1, \ldots, \rho_n), a_1, \ldots, a_n)$. Let s_i be a Coxeter generator. Let I denote the $|C_T| \times |C_T|$ identity matrix and let $M_{i,T}$ denote the matrix of τ_{i,V_T} with respect to the basis C_T. Set $r = \frac{(a_{i+1}-a_i) dim(V^{\rho_i})}{|G|}$ and $M = M_{i,T}$.*

The action of s_i on V_T is as follows.

(i) If i and $i+1$ are in the same column of the same Young diagram of T then V_T is closed under the action of s_i and the matrix of this action with respect to the basis C_T is M.

(ii) If i and $i+1$ are in the same row of the same Young diagram of T then V_T is closed under the action of s_i and the matrix of this action with respect to the basis C_T is $-M$.

(iii) Suppose i and $i+1$ are not in the same Young diagram of T. Let $S = s_i \cdot T$. Then $V_T \oplus V_S$ is closed under the action of s_i and the matrix of this action, with respect to the basis $C_T \cup C_S$, is given by

$$\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}.$$

(iv) Suppose i and $i+1$ are in the same Young diagram of T but not in the same row or same column of this Young diagram. Let $S = s_i \cdot T$. Then $M = M_{i,S}$.

Then $V_T \oplus V_S$ is closed under the action of s_i and the matrix of this action, with respect to the basis $C_T \cup C_S$, is given by

$$\begin{bmatrix} r^{-1}M & \sqrt{1-r^{-2}}\, I \\ \sqrt{1-r^{-2}}\, I & -r^{-1}M \end{bmatrix}.$$

7 Generalized Johnson scheme

The simplest nontrivial examples of the Vershik-Okounkov theory are the classical "Johnson schemes" and the "generalized Johnson schemes" of Ceccherini-Silberstein, Scarabotti,

and Tolli [**1, 2**] (also see [**8**]). We consider multiplicity free S_n, G_n-actions and explicitly write down the GZ-vectors (in the S_n case) and the GZ-subspaces (in the G_n case) and also identify the irreducibles which occur.

We begin with the S_n action. Let $B(n)$ denote the set of all subsets of $[n] = \{1, 2, \ldots, n\}$ and, for $0 \leq i \leq n$, let $B(n)_i$ denote the set of all subsets of $[n]$ with cardinality i. There is a natural action of S_n on $B(n)_i$ and $B(n)$. For a finite set S, let $V(S)$ denote the complex vector space with S as basis.

We have the following direct sum decomposition into S_n-submodules of the permutation representation of S_n on $V(B(n))$:

$$V(B(n)) \quad = \quad V(B(n)_0) \oplus V(B(n)_1) \oplus \cdots \oplus V(B(n)_n). \tag{33}$$

The following result is classical ([**2, 12**]). We give a constructive proof that produces an explicit Gelfand-Tsetlin basis.

Theorem 7.1 *For $0 \leq i \leq n$, $V(B(n)_i)$ is a multiplicity free S_n-module with S_n-module isomorphism*

$$V(B(n)_i) \ \cong \ \bigoplus_k V^{(n-k,k)},$$

where the sum is over all partitions $(n-k, k)$ of n with atmost two parts satisfying $k \leq i \leq n-k$.

An element $v \in V(B(n))$ is *homogeneous* if $v \in V(B(n)_k)$ for some k. We say that a nonzero homogeneous element v is of *rank* k, and we write $r(v) = k$, if $v \in V(B(n)_k)$. The *up operator* $U_n : V(B(n)) \to V(B(n))$ is defined, for $X \in B(n)$, by

$$U_n(X) = \sum_Y Y,$$

where the sum is over all $Y \in B(n)$ covering X, i.e., $X \subseteq Y$ and $|Y| = |X| + 1$.

A *symmetric Jordan chain* (SJC) in $V(B(n))$ is a sequence $v = (v_1, \ldots, v_h)$ of nonzero homogeneous elements of $V(B(n))$ such that $U_n(v_{i-1}) = v_i$, for $i = 2, \ldots h$, $U_n(v_h) = 0$, and $r(v_1) + r(v_h) = n$, if $h \geq 2$, or else $2r(v_1) = n$, if $h = 1$. Note that the elements of the sequence v are linearly independent, being nonzero and of different ranks. We say that v *starts* at rank $r(v_1)$ and *ends* at rank $n - r(v_1)$. We do not distinguish between the sequence $(v_1, \ldots, v_h)$ and the underlying set $\{v_1, \ldots, v_h\}$. A *symmetric Jordan basis* (SJB) of $V(B(n))$ is a basis of $V(B(n))$ consisting of a disjoint union of SJC's in $V(B(n))$. Given an SJB $J(n)$ of $V(B(n))$ and $0 \leq k \leq n/2$, let $\mathcal{J}(n,k)$ denote the set of all SJC's in $J(n)$ starting at rank k and ending at rank $n-k$ and let $J(n,k)$ denote the union of all SJC's in $\mathcal{J}(n,k)$.

Given $T \in \text{tab}(n, \mu)$, where μ has atmost two rows, we denote by $T +_1 (n+1)$ the standard Young tableaux obtained from T by adding $n+1$ at the end of the first

row. Similarly, given $T \in \text{tab}(n, \mu)$, where μ has atmost two rows with the second row containing fewer elements than the first row, we denote by $T +_2 (n+1)$ the standard Young tableaux obtained from T by adding $n+1$ at the end of the second row. The basic idea of the following algorithm is from **[10]**, though we have added new elements here, namely, Theorems 7.4 and 7.5.

Theorem 7.2 *There exists an inductive procedure to explicitly construct an SJB $J(n)$ of $V(B(n))$ and, for $0 \leq k \leq n/2$, a bijection*

$$B_{n,k} : tab(n, (n-k, k)) \to \mathcal{J}(n, k). \tag{34}$$

Proof The case $n = 1$ is clear.

Let $V = V(B(n+1))$. Define $V(0)$ to be the subspace of V generated by all subsets of $[n+1]$ not containing $n+1$ and define $V(1)$ to be the subspace of V generated by all subsets of $[n+1]$ containing $n+1$. We have $V = V(0) \oplus V(1)$. The linear map $\mathcal{R} : V(0) \to V(1)$, given by $X \mapsto X \cup \{n+1\},\ X \subseteq [n]$ is an isomorphism. We write $\mathcal{R}(v) = \overline{v}$. We write U for the up operator U_{n+1} on V and we write U_0 for the up operator on $V(0)$ $(= V(B(n)))$. We have, for $v \in V(0)$,

$$U(v) = U_0(v) + \overline{v},\ \ U(\overline{v}) = \overline{U_0(v)}. \tag{35}$$

By induction hypothesis there is an SJB $J(n)$ of $V(B(n)) = V(0)$ and bijections $B_{n,k}$ as in (34) above. We shall now produce an SJB $J(n+1)$ of V by producing, for each SJC in $J(n)$, either one or two SJC's in V such that the collection of all these SJC's is a basis.

Let $0 \leq k \leq n/2$. Consider $T \in \text{tab}(n, (n-k, k))$ and consider the SJC $B_{n,k}(T) = (x_k, \ldots, x_{n-k}) \in \mathcal{J}(n, k)$, where $r(x_k) = k$.

We now consider two cases.

(a) $k = n - k$: From (35) we have $U(x_k) = \overline{x_k}$ and $U(\overline{x_k}) = \overline{U_0(x_k)} = 0$. Since $\mathcal{R}$ is an isomorphism $\overline{x_k} \neq 0$. Define

$$B_{n+1,k}(T +_1 (n+1)) \ \ = \ \ (x_k, \overline{x_k}). \tag{36}$$

(b) $k < n - k$: Set $x_{k-1} = x_{n+1-k} = 0$ and define

$$y = (y_k, \ldots, y_{n+1-k}),\ \text{and}\ z = (z_{k+1}, \ldots, z_{n-k}), \tag{37}$$

by

$$y_l \ = \ x_l + (l-k)\,\overline{x_{l-1}},\ \ k \leq l \leq n+1-k. \tag{38}$$

$$z_l \ = \ (n-k-l+1)\,\overline{x_{l-1}} - x_l,\ \ k+1 \leq l \leq n-k. \tag{39}$$

From (35) we have

$$U(\overline{x_l}) = \overline{U_0(x_l)} = \overline{x_{l+1}},\ k \leq l \leq n-k \tag{40}$$

It thus follows from (35) and (40) that, for $k \leq l < n+1-k$, we have

$$U(y_l) = U(x_l + (l-k)\overline{x_{l-1}}) = x_{l+1} + \overline{x_l} + (l-k)\overline{x_l} = x_{l+1} + (l-k+1)\overline{x_l} = y_{l+1}.$$

Note that when $l = k$ the second step above is justified because of the presence of the $(l-k)$ factor even though $U(\overline{x_{k-1}}) = 0 \neq \overline{x_k}$. We also have $U(y_{n+1-k}) = U((n+1)\overline{x_{n-k}}) = (n+1)\overline{U_0(x_{n-k})} = 0$.

Similarly, for $k+1 \leq l < n-k$, we have

$$U(z_l) = U((n-k-l+1)\overline{x_{l-1}} - x_l) = (n-k-l+1)\overline{x_l} - x_{l+1} - \overline{x_l} = (n-k-l)\overline{x_l} - x_{l+1} = z_{l+1}.$$

and $U(z_{n-k}) = U(\overline{x_{n-k-1}} - x_{n-k}) = \overline{x_{n-k}} - \overline{x_{n-k}} = 0$.

Since $y_k = x_k \neq 0$, $y_{n+1-k} = (n+1)\overline{x_{n-k}} \neq 0$, x_l and $\overline{x_{l-1}}$ are linearly independent, for $k+1 \leq l \leq n-k$ and the 2×2 matrix

$$\begin{pmatrix} 1 & l-k \\ -1 & n-k-l+1 \end{pmatrix}$$

is nonsingular for $k+1 \leq l \leq n-k$, it follows that (37) gives two independent SJC's in V. Define

$$\begin{aligned} B_{n+1,k}(T +_1 (n+1)) &= y, \\ B_{n+1.k+1}(T +_2 (n+1)) &= z, \end{aligned}$$

and set $J(n+1)$ to be the union of all SJC's obtained in steps (36) and (37) above.

Since $V = V(0) \oplus V(1)$ and $\mathcal{R}$ is an isomorphism it follows that $J(n+1)$ is an SJB of V. That the maps $B_{n+1,k}$ are bijections is also clear. $\square$

Example 7.3 In this example we work out the SJB's of $V(B(n))$, for $n = 2, 3$, starting with the SJB of $V(B(1))$, using the formulas (36, 37, 38, 39) given in the proof of Theorem 7.2.

(i) The SJB of $V(B(1))$ is given by

$$(\ \emptyset\ ,\ \{1\}\ )$$

(ii) The SJB of $V(B(2))$ consists of

$$\begin{gathered} (\ \emptyset\ ,\ \{1\} + \{2\}\ ,\ 2\{1,2\}\ ) \\ (\ \{2\} - \{1\}\ ) \end{gathered}$$

(iii) The SJB of $V(B(3))$ consists of

$$\begin{gathered} (\ \emptyset\ ,\ \{1\} + \{2\} + \{3\}\ ,\ 2(\{1,2\} + \{1,3\} + \{2,3\})\ ,\ 6\{1,2,3\}\ ) \\ (\ 2\{3\} - \{1\} - \{2\}\ ,\ \{1,3\} + \{2,3\} - 2\{1,2\}\ ) \\ (\ \{2\} - \{1\}\ ,\ \{2,3\} - \{1,3\}\ ) \end{gathered}$$

For $0 \le k \le i \le n-k \le n$ define

$$J(n,k,i) \;=\; \{v \in J(n,k) : r(v) = i\}.$$

Let $W(n,k,i)$ be the subspace of $V(B(n)_i)$ spanned by $J(n,k,i)$. Then we have the direct sum decomposition

$$V(B(n)_i) \;=\; \bigoplus_{k=0}^{\min\{i,n-i\}} W(n,k,i), \;\; 0 \le i \le n. \tag{41}$$

We claim that each $W(n,k,i)$ is a S_n-submodule of $V(B(n)_i)$. We prove this by induction on i, the case $i=0$ being clear. Assume inductively that $W(n,0,i-1),\ldots,W(n,i-1,i-1)$ are submodules, where $i < \lfloor n/2 \rfloor$. Since U_n is S_n-linear, $U_n(W(n,j,i-1)) = W(n,j,i)$, $0 \le j \le i-1$ are submodules. Now consider $W(n,i,i)$. Let $u \in W(n,i,i)$ and $\pi \in S_n$. Since U_n is S_n-linear we have $U_n^{n+1-2i}(\pi u) = \pi U_n^{n+1-2i}(u) = 0$. It follows that $\pi u \in W(n,i,i)$. So the claim is proven for $0 \le i \le n/2$ and it follows for $i > n/2$ since U_n is S_n-linear.

Theorem 7.4 *As S_n-modules we have*

$$W(n,k,i) \;\cong\; V^{(n-k,k)}, \;\; 0 \le k \le i \le n-k \le n. \tag{42}$$

Proof By induction on n. The cases $n = 1,2,3$ can be directly verified from Example 7.3 (the main point to check is that $W(3,1,1)$ is the standard representation of S_3).

Now assume we have proven the result upto $n \ge 3$. By the algorithm of Theorem 7.2 we have, for $0 \le i \le n+1$,

$$W(n+1,k,i) \;=\; W(n,k,i) \oplus \overline{W}(n,k-1,i-1), \tag{43}$$

where $\overline{W} = \{\overline{v} \mid v \in W(n,k-1,i-1)\}$ (in the notation used in the proof of Theorem 7.2) and where $W(n,k,i)$ is taken to be the zero subspace if $i < k$ or $i > n-k$.

Now, $W(n+1,k,i)$ is a S_{n+1}-module and it is easily seen that $W(n,k,i)$ and $\overline{W}(n,k-1,i-1)$ are S_n-submodules of $W(n+1,k,i)$. By induction hyphothesis we have, as S_n-modules,

$$W(n,k,i) \;=\; V^{(n-k,k)}, \tag{44}$$

$$\overline{W}(n,k-1,i-1) \;=\; V^{(n-k+1,k-1)}. \tag{45}$$

Suppose an S_{n+1}-irreducible V^{λ}, where the Young diagram λ has 3 or more rows, occurs in $W(n+1,k,i)$. Since $n+1 \ge 4$, it follows that λ has an inner corner whose removal still leaves 3 or more rows. By the branching rule this contradicts (43), (44), and (45). So, for any S_{n+1}-irreducible V^{λ} occuring in $W(n+1,k,i)$, there are atmost two rows in λ. It is now easy to see using the branching rule and (44) and (45) that $W(n+1,k,i) \cong V^{(n+1-k,k)}$.
$\square$

Theorem 7.1 now follows from (41) and Theorem 7.4. We also have that

$$\dim(V^{(n-k,k)}) = \binom{n}{k} - \binom{n}{k-1}. \tag{46}$$

Summing (41) over i and taking dimensions we get

$$2^n = \sum_{k=0}^{\lfloor n/2 \rfloor} (n-2k+1) \left\{ \binom{n}{k} - \binom{n}{k-1} \right\}. \tag{47}$$

We denote the YJM elements of S_n by $Y_1, \ldots, Y_n$.

Theorem 7.5 *For $T \in tab(n,(n-k,k))$ and every vector v in the SJC $B_{n,k}(T)$ we have*

$$Y_j(v) = c(b_T(j))v, \; j = 1, 2, \ldots, n. \tag{48}$$

Proof We first show inductively that each element of $J(n)$ is a simultaneous eigenvector of $Y_1, \ldots, Y_n$, the case $n = 1$ being clear.

Note that if $v \in V(B(n)_k)$ is an eigenvector for Y_i, for some $1 \leq i \leq n$, then $\overline{v} \in V(B(n+1)_{k+1})$ is also an eigenvector for Y_i with the same eigenvalue. Thus it follows from (36, 37, 38, 39) that each element of $J(n+1)$ is an eigenvector for $Y_1, \ldots, Y_n$. It remains to show that each element of $J(n+1)$ is an eigenvector for Y_{n+1}.

We now have from Theorem 7.4 that, for $0 \leq i \leq \frac{n+1}{2}$, $W(n+1,0,i), \ldots, W(n+1,i,i)$ are mutually nonisomorphic irreducibles. Consider the S_{n+1}-linear map $f : V(B(n+1)_i) \rightarrow V(B(n+1)_i)$ given by $f(v) = av$, where

$$a = \text{ sum of all transpositions in } S_{n+1} = Y_1 + \cdots + Y_{n+1}.$$

It follows by Schur's lemma that there exist scalars $\alpha_0, \ldots, \alpha_i$ such that $f(u) = \alpha_k u$, for $u \in W(n+1,k,i)$. Thus each element of $J(n+1,k,i)$ is an eigenvector for $Y_1 + \cdots + Y_{n+1}$ (and also for $Y_1, \ldots, Y_n$). It follows that each element of $J(n+1,k,i)$ is an eigenvector for Y_{n+1}.

The paragraph above has shown that the first element of each symmetric Jordan chain in $J(n+1)$ is a simultaneous eigenvector for $Y_1, \ldots, Y_{n+1}$. It now follows (since U_{n+1} is S_{n+1}-linear) that each element of $J(n+1)$ is a simultaneous eigenvector for $Y_1, \ldots, Y_{n+1}$.

We are left to show that, for each $v \in B_{n,k}(T)$, the eigenvalues of $Y_1, \ldots, Y_n$ on v are given by (48). We can show this by induction, the case $n = 1$ being trivial.

Just like above the eigenvalues of $Y_1, \ldots, Y_n$ on $v \in B_{n+1,k}(T)$ will continue to satisfy (48). Now, since v is an eigenvector for Y_{n+1} and v lies in an S_{n+1}-irreducible isomorphic to $V^{(n+1-k,k)}$, it follows that the eigenvalue of Y_{n+1} on v also satisfies (48). That completes the proof. □

See [**4**] for an elegant direct construction of the GZ-basis given above and see [**5**] for an application to complexity theory.

Now we study the G_n analog of the S_n-action considered above, defined in [**1, 2**]. Let G be a finite group acting on the finite set X. Assume that the corresponding permutation representation on $V(X)$ is multiplicity free. This implies, in particular, that the action is transitive.

Let L_0 be a symbol not in X and let Y denote the *alphabet* $Y = \{L_0\} \cup X$. We call the elements of X the *nonzero* letters in Y. Define $B_X(n) = \{(a_1, \ldots, a_n) : a_i \in Y \text{ for all } i\}$, the set of all n-tuples of elements of Y (we use L_0 instead of 0 for the zero letter for later convenience. We do not want to confuse the letter 0 with the vector 0). Given $a = (a_1, \ldots, a_n) \in B_X(n)$, define the *support* of a by $S(a) = \{i \in [n] : a_i \neq L_0\}$. For $0 \leq i \leq n$, $B_X(n)_i$ denotes the set of all elements $a \in B_X(n)$ with $|S(a)| = i$. We have

$$|B_X(n)| = (|X|+1)^n, \quad |B_X(n)_i| = \binom{n}{i}|X|^i.$$

(We take the binomial coefficient $\binom{n}{k}$ to be 0 if $n < 0$ or $k < 0$).

There is a natural action of the wreath product G_n on $B_X(n)$ and $B_X(n)_i$: permute the n coordinates followed by independently acting on the nonzero letters by elements of G. In detail, given $(g_1, g_2, \ldots, g_n, \pi) \in G_n$ and $a = (a_1, \ldots, a_n) \in B_X(n)$, we have $(g_1, \ldots, g_n, \pi)(a_1, \ldots, a_n) = (b_1, \ldots, b_n)$, where $b_i = g_i a_{\pi^{-1}(i)}$, if $a_{\pi^{-1}(i)}$ is a nonzero letter and $b_i = L_0$, if $a_{\pi^{-1}(i)} = L_0$. We have the following direct sum decomposition into G_n-submodules of the permutation representation of G_n on $V(B_X(n))$:

$$V(B_X(n)) \;=\; V(B_X(n)_0) \oplus V(B_X(n)_1) \oplus \cdots \oplus V(B_X(n)_n). \tag{49}$$

We now introduce some notation. Let $\sigma_1, \ldots, \sigma_m$, $\sigma_i \in G^\wedge$, be the distinct irreducible G-representations occuring in the multiplicity free G-module $V(X)$. We assume that σ_1 is the trivial representation. Now enumerate all the elements of $G^\wedge$ as $\sigma_1, \ldots, \sigma_t$, so that $\sigma_{m+1}, \ldots, \sigma_t$ do not appear in $V(X)$. For $i = 1, \ldots, m$, set $d_i = \dim(V^{\sigma_i})$, so that $d_1 = 1$ and $d_1 + \cdots + d_m = |X|$.

Denote by $\mathcal{Y}_{2,n}(G^\wedge)$ the set of all $\mu \in \mathcal{Y}_n(G^\wedge)$ such that

(i) $\mu(\sigma_i)$ is the empty partition, for $i = m+1, \ldots, t$.

(ii) $\mu(\sigma_i)$ has atmost one part, denoted $p_i(\mu)$, for $i = 2, \ldots, m$. We have $p_i(\mu) = 0$ if $\mu(\sigma_i)$ is the empty partition. We set $s(\mu) = p_2(\mu) + \cdots + p_m(\mu)$.

(iii) $\mu(\sigma_1)$ has atmost two parts, denoted $a(\mu), b(\mu)$, with $a(\mu) \geq b(\mu)$. Just like in item (ii) above, one or both of $a(\mu), b(\mu)$ may be 0.

We have the following combinatorial identity (recall that V^μ denotes the irreducible G_n-module parametrized by $\mu \in \mathcal{Y}_n(G^\wedge)$).

Theorem 7.6 *We have*

$$(|X|+1)^n \;=\; \sum_{\mu \in \mathcal{Y}_{2,n}(G^\wedge)} (1 + a(\mu) - b(\mu))\ \dim(V^\mu). \tag{50}$$

Proof The proof is in two steps.

(a) Let $C(n,m)$ denote the set of all m-tuples of nonnegative integers with sum n. We have, using the multinomial theorem and (47) above,

$$\begin{aligned}
&(|X|+1)^n \\
&= (d_1+d_2+\cdots+d_m+1)^n \\
&= (d_2+\cdots+d_m+2)^n \\
&= \sum_{(p_1,\ldots,p_m)\in C(n,m)} \binom{n}{p_1,\ldots,p_m} d_2^{p_2}\cdots d_m^{p_m}\, 2^{p_1} \\
&= \sum_{(p_1,\ldots,p_m)\in C(n,m)} \sum_{k=0}^{\lfloor p_1/2\rfloor} (p_1-2k+1) \binom{n}{p_1,\ldots,p_m} d_2^{p_2}\cdots d_m^{p_m} \left\{ \binom{p_1}{k} - \binom{p_1}{k-1} \right\}.
\end{aligned}$$

(b) Let $\mu \in \mathcal{Y}_{2,n}(G^\wedge)$. We have

$$V^\mu \;=\; \oplus_{T\in \mathrm{tab}_{G(n,\mu)}} V_T.$$

The dimension of the GZ-subspace V_T of V^μ is clearly $d_2^{p_2(\mu)}\cdots d_m^{p_m(\mu)}$. With μ bijectively associate the pair of elements

$$(a(\mu)+b(\mu), p_2(\mu),\ldots,p_m(\mu)) \in C(n,m) \text{ and } b(\mu)\in\mathbb{N} \text{ with } b(\mu) \le \lfloor (a(\mu)+b(\mu))/2 \rfloor.$$

It is easy to see, using (46) above, that the cardinality of $\mathrm{tab}_G(n,\mu)$ is

$$\binom{n}{a(\mu)+b(\mu), p_2(\mu)\ldots,p_m(\mu)} \left\{ \binom{a(\mu)+b(\mu)}{b(\mu)} - \binom{a(\mu)+b(\mu)}{b(\mu)-1} \right\}.$$

The result now follows from steps (a) and (b) above. $\Box$

We shall now give a representation theoretic interpretation to Theorem 7.6 above.

Consider the tensor product

$$\otimes_{i=1}^n V(Y) = V(Y)\otimes\cdots\otimes V(Y) \;\; (n \text{ factors}),$$

with the natural G_n-action (permute the factors and then independently act on the factors by elements of G). There is a G_n-linear isomorphism

$$V(B_X(n)) \;\cong\; \otimes_{i=1}^n V(Y) \tag{51}$$

given by $a=(a_1,\ldots,a_n)\mapsto a_1\otimes\cdots\otimes a_n,\; a\in B_X(n)$. From now onwards, we shall not distinguish between $V(B_X(n))$ and $\otimes_{i=1}^n V(Y)$. The image of $V(B_X(n)_i)$ is denoted $(\otimes_{i=1}^n V(Y))_i$.

Consider the canonical decomposition

$$V(X) = W_1 \oplus\cdots\oplus W_m,$$

of $V(X)$ into distinct irreducible G-submodules, where W_i is isomorphic to V^{σ_i}, for $1 \leq i \leq m$. Thus $d_i = \dim W_i$, $i = 1, \ldots, m$.

Define the vector $z \in V(Y)$ by $z = \sum_{x \in X} x$.

For $0 \leq i \leq n$ set

$$\mathcal{Y}_{2,n}(G^\wedge)_i \;=\; \{\mu \in \mathcal{Y}_{2,n}(G^\wedge) \mid b(\mu) + s(\mu) \leq i \leq a(\mu) + s(\mu)\}\,.$$

Theorem 7.7 *For $0 \leq i \leq n$, $V(B_X(n)_i)$ is a multiplicity free G_n-module with G_n-module isomorphism*

$$V(B_X(n)_i) \;\cong\; \bigoplus_{\mu \in \mathcal{Y}_{2,n}(G^\wedge)_i} V^\mu.$$

Proof Let $\mu \in \mathcal{Y}_{2,n}(G^\wedge)$ and $a(\mu) + s(\mu) \leq i \leq b(\mu) + s(\mu)$. Let $R \in \mathrm{tab}_G(n, \mu)$ be as defined in the proof of Lemma 6.2. We shall exhibit a GZ-subspace W of $(\otimes_{j=1}^n V(Y))_i$ of type V_R, i.e, W is closed under the G^n-action and, as a G^n-module, is isomorphic to $V^{r_R(1)} \otimes \cdots \otimes V^{r_R(n)}$ and, for $v \in W$ and $j = 1, 2, \ldots, n$, we have

$$X_j(v) \;=\; \frac{|G|}{\dim(V^{r_R(j)})}\, c(b_R(j))v. \tag{52}$$

This will show that V^μ appears in $V(B_X(n)_i)$. The dimension count given by Theorem 7.6 then completes the proof.

(a) Set $q = a(\mu) + b(\mu)$. There is an injection

$$\Gamma : V(B(q)) \to \otimes_{j=1}^q V(Y)$$

given as follows: for $X \subseteq [q]$, we have $\Gamma(X) = u_1 \otimes \cdots \otimes u_q$, where $u_k = L_0$, if $k \notin X$ and $u_k = z$, if $k \in X$.

Since $b(\mu) \leq i - s(\mu) \leq a(\mu)$ and $b(\mu) \leq \lfloor (a(\mu) + b(\mu))/2 \rfloor$, it follows from Theorem 7.1 that there is a vector $u \in V(B(q)_{i-s(\mu)})$ (determined uniquely upto scalars) such that

$$Y_j(u) \;=\; c(b_R(j))u,\; j = 1, \ldots, q. \tag{53}$$

(b) Let $\sigma \in G^\wedge$ and consider the G_k-module $V^\sigma \otimes \cdots \otimes V^\sigma$ (k factors). It follows from Theorem 5.3(i) and (ii)(a) that, for all $v \in V^\sigma \otimes \cdots \otimes V^\sigma$,

$$X_j(v) = (j-1)\frac{|G|}{\dim(V^\sigma)}v,\; j = 1, \ldots, k. \tag{54}$$

Consider the subspace W of $(\otimes_{j=1}^n V(Y))_i$ given by

$$W = \mathrm{Span}(\Gamma(u)) \otimes W_2 \otimes \cdots \otimes W_2 \otimes W_3 \otimes \cdots \otimes W_3 \otimes \cdots \otimes W_m \otimes \cdots \otimes W_m,$$

where W_2 is repeated $p_2(\mu)$ times, W_3 is repeated $p_3(\mu)$ times, and so on until W_m is repeated $p_m(\mu)$ times.

Since $g \cdot L_0 = L_0$ and $g \cdot z = z$, for all $g \in G$, it follows that W is closed under the G^n-action and, as a G^n-module, is isomorphic to $V^{r_R(1)} \otimes \cdots \otimes V^{r_R(n)}$. Moreover, it follows from (53) above that, for $v \in W$,

$$X_j(v) \;=\; \frac{|G|}{\dim(V^{r_R(j)})}\, c(b_R(j))v,\; j = 1, \ldots, q.$$

From (54) above and Theorem 5.3(v) we see that, for $v \in W$,

$$X_j(v) \;=\; \frac{|G|}{\dim(V^{r_R(j)})}\, c(b_R(j))v,\; j = q+1, \ldots, n.$$

That completes the proof. $\square$

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